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Central Limit Theorems and Invariance Principles for Time-One Maps of Hyperbolic Flows*

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Abstract: We give a general method for deducing statistical limit laws in situations where rapid decay of correlations has been established. As an application of this method, we obtain new results for time-one maps of hyperbolic flows.

In particular, using recent results of Dolgopyat, we prove that many classical limit theorems of probability theory, such as the central limit theorem, the law of the iterated logarithm, and approximation by Brownian motion (almost sure invariance principle), are typically valid for such time-one maps.

The central limit theorem for hyperbolic flows goes back to Ratner 1973 and is always valid, irrespective of mixing hypotheses.

1. Introduction

Let $\Lambda \subset M$ be a topologically mixing hyperbolic basic set for a smooth flow T_t on a compact manifold M. Let μ denote an equilibrium measure supported on Λ, corresponding to a Hölder continuous potential [7]. In this paper, we are interested in proving statistical limit laws such as the central limit theorem for the time-one map $T = T_1$ of such a flow.

We note that such limit laws are well-known for the hyperbolic flow itself. See Ratner [22] for the central limit theorem, Wong [28] for the law of the iterated logarithm, and Denker and Philipp [9] for the almost sure invariance principle. See also [18].

The validity of such results for time-one maps is considerably more delicate than that for flows. To see this, suppose that X is a mixing hyperbolic basic set and $r : X \to \mathbb{R}$ is a Hölder roof function. Let X_r denote the suspension of X and consider the suspension flow $T_t : X_r \to X_r$. Suppose that r is cohomologous to

* This differs from the published version: §4 was deleted, because the example given there, of a mixing but not rapidly mixing suspension, was incorrect.

a rational constant (for example, take $r \equiv 1$). Then the time-one map $T = T_1$ is far from ergodic and the above statistical limit laws fail abjectly. Nevertheless, these results are valid for the flow [18].

Dolgopyat [10] gave necessary and sufficient conditions for hyperbolic flows to exhibit rapid decay of correlations in the sense that for each $n \geq 1$, and all sufficiently regular observations $\phi, \psi : \Lambda \to \mathbb{R}$, there exists a constant $C(\phi, \psi, n)$ such that

$$\left|\int \phi(\psi \circ T_t)\, d\mu - \int \phi\, d\mu \int \psi\, d\mu\right| \leq C(\phi, \psi, n)/|t|^n, \tag{1}$$

for all $t \in \mathbb{R}$. Dolgopyat also proved that a sufficient condition for this result to hold is that there are periodic points $x_1, x_2 \in \Lambda$ with periods P_1, P_2 such that P_1/P_2 is Diophantine. Thus most hyperbolic flows are rapidly mixing (whereas previously Ruelle [24] and Pollicott [21] had proved the existence of mixing hyperbolic flows whose rates of mixing are arbitrarily slow).

An important feature of this theorem is that, for fixed ϕ, condition (1) holds for a large class of "test functions" ψ. Indeed, as a first step, Dolgopyat proves this result for one-sided subshifts where ψ is required only to be L^∞ and $C(\phi, \psi, n) = D(\phi, n)|\psi|_\infty$.

In this paper, we prove that a simple consequence of such an "L^∞" rapid decay result is that any sufficiently regular mean zero observation ϕ is cohomologous in L^p to a martingale for all $p \in [2, n)$. Here, $n > 4$ is sufficiently rapid decay for our purposes (and $n > 2$ suffices for the CLT).

As a consequence of the martingale reduction, we derive several classical limit theorems, the most powerful being the *almost sure invariance principle.*

Theorem 1. *Let $\Lambda \subset M$ be a topologically mixing hyperbolic basic set for a smooth flow T_t with equilibrium measure μ, corresponding to a Hölder continuous potential. Suppose that there are periodic points $x_1, x_2 \in \Lambda$ with periods P_1, P_2 such that P_1/P_2 is Diophantine. Let $\phi : M \to \mathbb{R}$ be sufficiently regular* [1] *with mean zero ($\int \phi\, d\mu = 0$) and $\int_0^t \phi \circ T_s\, ds$ unbounded. Then there is a Brownian motion W with variance*

$$\sigma^2 = \lim_{N\to\infty} \frac{1}{N} \int_\Lambda \Big(\sum_{j=0}^{N-1} \phi \circ T_j\Big)^2 d\mu > 0,$$

and a sequence of random variables $\{S(N) : N \geq 1\}$, equal in distribution to the sequence $\{\sum_{j=0}^{N-1} \phi \circ T_j : N \geq 1\}$, such that for each $\delta > 0$,

$$S([t]) = W(t) + O(t^{1/4+\delta}) \qquad \textit{as } t \to \infty,$$

almost surely.

Remark 1. The ASIP for flows (with $\sum_{j=0}^{N-1} \phi \circ T_j$ replaced by $\int_0^N \phi \circ T_t\, dt$) is an immediate consequence of the ASIP for time-one maps, since $\int_0^1 \phi \circ T_t\, dt$ satisfies the hypotheses of Theorem 1. As mentioned earlier, the ASIP for hyperbolic flows is valid even when mixing fails [9, 18].

[1] it suffices that ϕ is C^∞ in the flow direction, and that ϕ together with its time derivatives are Hölder continuous for some fixed Hölder exponent

Consequences of the ASIP include the central limit theorem, the weak invariance principle and the law of the iterated logarithm, see [20, 12].

We note that Dolgopyat [11], using rather different methods, has proved a version of the above result for time-one maps of Anosov flows with jointly non-integrable stable and unstable foliations.

Remark 2. The error term $O(t^{1/4+\delta})$ for all $\delta > 0$ improves the error term $O(t^{1/2-\alpha})$ for some $\alpha < 0$ which is more usual in the literature [9, 11, 20]. The improved error term is obtained also in [12, 18].

In Section 2, we prove a simple (but apparently novel) abstract result relating rapid mixing and approximation by a martingale. The central limit theorem and weak invariance principle for suspensions of one-sided subshifts of finite type are then an immediate consequence of Dolgopyat's rapid mixing theorem. In Section 3, we prove Theorem 1 by passing in the standard way from one-sided subshifts to two-sided subshifts [25, 6] and then from suspensions of two-sided subshifts to hyperbolic flows [5].

2. Decay of correlations and martingales

In this section we prove a simple result that derives statistical limit theorems such as the central limit theorem as a consequence of rapid decay of correlations.

Proposition 1. *Let (Y, m) be a probability space and $T : Y \to Y$ be a measure preserving transformation. Let $f \in L^\infty$. Suppose that there exists a constant $C > 0$ such that*

$$|\textstyle\int_Y f\ (g \circ T)\, dm| \le C|g|_\infty,$$

for all $g \in L^\infty$. Define $Ug = g \circ T$, so $U : L^p \to L^p$ is an isometry for all $1 \le p \le \infty$. Let $U^* : L^2 \to L^2$ be the L^2-adjoint of U.

Then $U^* f \in L^\infty$ and $|U^* f|_p \le C^{1/p}|f|_\infty^{(p-1)/p}$ for all $p \ge 1$ finite, $|U^* f|_\infty \le |f|_\infty$.

Proof. By assumption, we have

$$|\textstyle\int (U^* f)\ g| = |\int f\ Ug| \le C|g|_\infty.$$

By duality, $|U^* f|_1 \le C$. (Take $g = \operatorname{sgn}(U^* f)$.)

Next we derive the L^∞ estimate. Let $\epsilon > 0$ and suppose that $|U^* f| \ge |f|_\infty + \epsilon$ on a set A. Take $g = \chi_A\ \operatorname{sgn}(U^* f)$. Then

$$\mu(A)[|f|_\infty + \epsilon] \le |\textstyle\int (U^* f)\ g| \le \int |f\ Ug| = \int_{T^{-1}(A)} |f| \le \mu(T^{-1}(A))|f|_\infty = \mu(A)|f|_\infty,$$

so that $\mu(A) = 0$. Hence $|U^* f|_\infty \le |f|_\infty$.

Finally, compute that

$$\textstyle\int |U^* f|^p = \int |U^* f|^{p-1}|U^* f| \le |U^* f|_\infty^{p-1}|U^* f|_1 \le |f|_\infty^{p-1} C.$$

□

Lemma 1. *Let (Y, m) be a probability space and $T : Y \to Y$ be a measure preserving transformation. Define $U^* : L^2 \to L^2$ as in Proposition 1. Let $\phi : Y \to \mathbb{R}$ be in L^∞ with $\int_Y \phi\, dm = 0$.*

Fix $n > 2$, and suppose that there is a constant C (depending on ϕ and n) such that

$$\left|\int_Y \phi\, (\psi \circ T^j)\, dm\right| \le \frac{C}{j^n} |\psi|_\infty, \tag{2}$$

for all $\psi \in L^\infty$ and $j \ge 1$.

Then $\phi = \widehat{\phi} + \chi \circ T - \chi$ where $\widehat{\phi}$ and χ lie in L^p, for all $p < n$, and $U^* \widehat{\phi} = 0$.

Proof. It follows from Proposition 1 that $(U^*)^j \phi \in L^\infty$, and that

$$|(U^*)^j \phi|_p \le \frac{C^{1/p}}{j^{n/p}} |\phi|_\infty^{(p-1)/p}, \tag{3}$$

for all finite $p \ge 1$. If $p < n$, then $\sum_{j=1}^\infty (U^*)^j \phi$ converges absolutely in L^p. Define $\chi = \sum_{j=1}^\infty (U^*)^j \phi$ and $\widehat{\phi} = \phi - U\chi + \chi$. Then χ and $\widehat{\phi}$ lie in L^p. Moreover $U^* \widehat{\phi} = 0$ (cf. Gordin [13]). □

Remark 3. Assume that ϕ and $\widehat{\phi}$ are as in Lemma 1. Define $\phi_N = \sum_{j=0}^{N-1} U^j \phi$ and define $\widehat{\phi}_N$ similarly. Then $\phi_N = \widehat{\phi}_N + \chi \circ T^N - \chi$. If $\chi \in L^2$, then $\chi^2 \circ T^N = o(N)$ almost everywhere by Birkhoff's ergodic theorem, hence $\phi_N = \widehat{\phi}_N + o(N^{1/2})$ almost everywhere.

Theorem 2 (Central limit theorem (CLT)). *Let (Y, m) be a probability space and suppose that $T : Y \to Y$ is ergodic. Let $\phi : Y \to \mathbb{R}$ be in L^∞ with $\int_Y \phi\, dm = 0$. Suppose that ϕ satisfies condition (2) for some $n > 2$ (and all $\psi \in L^\infty$, $j \ge 1$). Then $\frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} \phi \circ T^j$ converges in distribution as $N \to \infty$ to a normal distribution with mean zero and variance σ^2 for some $\sigma \ge 0$.*

Moreover, $\sigma^2 = \lim_{N\to\infty} \frac{1}{N} \int_Y (\sum_{j=0}^{N-1} \phi \circ T^j)^2 dm$, and $\sigma^2 = 0$ if and only if ϕ is an L^p-coboundary for all $p < n$.

Proof. Choose $n > p \ge 2$ in Lemma 1 and Remark 3. Then $\phi_N = \widehat{\phi}_N + o(N^{1/2})$ so it suffices to prove the CLT with ϕ replaced by $\widehat{\phi}$. Passing to the natural extension [23], we obtain a biinfinite stationary ergodic martingale $\{X_j : j \in \mathbb{Z}\}$ where $X_{-j} = \widehat{\phi} \circ T^j$ for $j \ge 0$ (cf. [12, Remark 3.12]). Hence it follows from Billingsley [1] that $\frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} X_j$ converges to a normal distribution with mean zero and variance $\int X_1^2$ as $N \to \pm\infty$. In particular, $\frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} \widehat{\phi} \circ T^j$ converges to a normal distribution with mean zero and variance $\sigma^2 = \int \widehat{\phi}^2$. Moreover, the variance is zero if and only if $\widehat{\phi} = 0$ which means that $\phi = \chi \circ T - \chi$ is an L^p-coboundary.

Finally, we verify the formula for σ^2 in the last statement of the theorem. First note that $\sigma^2 = \int \widehat{\phi}^2 = \frac{1}{N} \int \widehat{\phi}_N^2$. That is, $\sigma = \frac{1}{\sqrt{N}} |\widehat{\phi}_N|_2$. Writing $\phi_N = \widehat{\phi}_N + \chi \circ T^N - \chi$, we compute that $|\phi_N|_2 \le |\widehat{\phi}_N|_2 + 2|\chi|_2$ so that $\limsup_{N\to\infty} \frac{1}{\sqrt{N}} |\phi_N|_2 \le \sigma$. Similarly, $\liminf_{N\to\infty} \frac{1}{\sqrt{N}} |\phi_N|_2 \ge \sigma$. □

Remark 4. Suppose that $T_t : Y \to Y$ is a semiflow and that the time-one map $T = T_1$ is ergodic and satisfies the rapid decay condition (2) for some $n > 2$. Then the conclusion of Theorem 2 is valid for the time-one map T. Moreover, replacing ϕ by $\int_0^1 \phi \circ T_t \, dt$, we conclude that $\frac{1}{\sqrt{T}} \int_0^T \phi \circ T_t \, dt$ converges in distribution as $T \to \infty$ to a normal distribution with mean zero and variance $\widetilde{\sigma}^2$ for some $\widetilde{\sigma} \geq 0$, and $\widetilde{\sigma}^2 = 0$ if and only if $\int_0^1 \phi \circ T_t \, dt$ is an L^p-coboundary.

Remark 5. Under the hypotheses of Theorem 2 (or Remark 4), the weak invariance principle (WIP) (otherwise known as the functional central limit theorem) follows by [2]. The use of martingale approximations to prove the CLT and WIP for dynamical systems is standard since Gordin [13]. In certain situations (see [8, 12] and Section 3 of the present paper) martingale approximation leads to the almost sure invariance principle and hence the law of the iterated logarithm. However, this step relies on the class of dynamical systems under consideration being closed under time-reversal (see [8, Remarques 2.7(1)] and [12, Remark 6.5]).

Remark 6. The key hypothesis in Theorem 2 is that for the fixed mean zero observation ϕ, the correlation function $\int_Y \phi(\psi \circ T^j) dm$ decays rapidly for *all* $\psi \in L^\infty$. Such a hypothesis cannot hold for an invertible mapping T, since the operator U^* appearing in the proof of the theorem would be a unitary operator and so could not be strictly contractive. However, this hypothesis is often satisfied when T is noninvertible.

We note that Theorem 2 is both more restricted and more general than a related result of Liverani [15]. Liverani requires only that $\int_Y \phi(\phi \circ T^j) dm$ decays rapidly (so $\psi = \phi$), and $n > 1$ is sufficiently rapid decay. However, Liverani requires an additional *a priori* estimate on the contractivity of the transfer operator.

Application to suspensions of one-sided subshifts of finite type. We recall the notion of a symbolic (semi)-flow [5,19]. Suppose that $\sigma : X^+ \to X^+$ is an aperiodic one-sided subshift of finite type. Fix $\theta \in (0,1)$. Define the metric $d_\theta(x,y) = \theta^N$ where N is the largest positive integer such that $x_i = y_j$ for all $i < N$. Define the Hölder space $F_\theta(X^+)$ consisting of continuous functions $v : X^+ \to \mathbb{R}$ that are Lipschitz with respect to this metric, with Lipschitz constant $|v|_\theta$. Let μ be an equilibrium measure on X^+ corresponding to a Hölder potential in $F_\theta(X^+)$.

Let $r \in F_\theta(X^+)$ be a strictly positive roof function, and define the suspension $X_r^+ = \{(x,s) \in X^+ \times \mathbb{R} : 0 \leq s \leq r(x)\}/\sim$ where $(x, r(x)) \sim (\sigma x, 0)$. The suspension (semi)-flow is given by $T_t(x,s) = (x, s+t)$ and the invariant measure $\mu_r = \mu \times \ell / \int r \, d\mu$ is an equilibrium measure for the flow, where ℓ is Lebesgue measure on $\mathbb{R}$.

Define the space $F_\theta(X_r^+)$ consisting of continuous functions $\phi : X_r^+ \to \mathbb{R}$ that are Lipschitz with respect to the metric $d_\theta(x,x') + |s - s'|$ on $X^+ \times \mathbb{R}$ restricted to $\{(x,s) \in X^+ \times \mathbb{R} : 0 \leq s \leq r(x)\}$. Note that the functions in $F_\theta(X_r^+)$ are continuous along the flow direction. Let $F_{k,\theta}(X_r^+)$ consist of functions ϕ that are C^k in the flow direction such that $\partial_t^j \phi \in F_\theta(X_r^+)$ for $j = 0, 1, \ldots, k$, and let $|\phi|_{k,\theta}$ denote the maximum of the Lipschitz constants corresponding to $\partial_t^j \phi$.

Theorem 3. *Let X_r^+ be a Hölder suspension of an aperiodic one-sided subshift of finite type, with Hölder equilibrium measure μ. Suppose that there are periodic points $y_1, y_2 \in X_r^+$ with periods P_1, P_2 such that P_1/P_2 is Diophantine. Then there is an integer $k \geq 1$ such that the CLT and WIP (for the time-one map as well as the flow) hold for all observations $\phi \in F_{k,\theta}(X_r^+)$ with $\int_{X_r} \phi \, d\mu_r = 0$.*

Proof. Under the Diophantine hypothesis, Dolgopyat [10] proved that for any $n \geq 1$, there exists an integer $k(n) \geq 1$ and a constant $C(n) > 0$ such that if $\phi \in F_{k(n),\theta}(X_r^+)$ and $\psi \in L^\infty(X_r^+)$, then

$$\left|\int \phi(\psi \circ T_t) d\mu_r - \int \phi d\mu_r \int \psi d\mu_r\right| \leq C(n)|\phi|_{k(n),\theta}|\psi|_\infty / t^n, \tag{4}$$

for all $t > 0$.

Take $n > 2$ in (4), and apply Theorem 2 and Remark 5. □

If the variance σ^2 vanishes (in Theorem 2), then the CLT and WIP (for $\frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} \phi \circ T^j$) are said to be degenerate. We want conditions that exclude this possibility. Similarly, in the CLT and WIP for $\frac{1}{\sqrt{T}} \int_0^T \phi \circ T_t \, dt$, (Remark 4) we wish to rule out the possibility that $\widetilde{\sigma}^2 = 0$. The next result shows that these situations are highly unlikely in the hyperbolic case.

Proposition 2. *Assume the set up of Theorem 3. The following are equivalent.*

(a) $\widetilde{\sigma}^2 = 0$,

(b) $\int_0^T \phi(T_s y) ds = 0$ *whenever y is a periodic point of period T,*

(c) There is a Hölder $g : X_r^+ \to \mathbb{R}$ such that $\int_0^t \phi \circ T_s \, ds = g - g \circ T_t$ for all t, and

(d) $\int_0^T \phi(T_s y) ds$ *is uniformly bounded (in $T > 0$ and $y \in X_r^+$).*

If $\sigma^2 = 0$, then conditions (a)–(d) hold.

Proof. The equivalence of (b) and (c) is the Livšic periodic point theorem [16], [14, Theorem 19.2.4]. It is clear that (c) implies (d). If (d) is valid, then the CLT is degenerate, so (d) implies (a).

If (a) is valid, then by Theorem 2, $\psi = \chi - \chi \circ T_1$ almost everywhere, where $\chi \in L^p$ $(2 \leq p < n)$ and $\psi = \int_0^1 \phi \circ T_u \, du$. Define $F_t = \int_0^t \psi \circ T_s \, ds$ and $h = \int_0^1 \chi \circ T_s \, ds$. Then $F_t : X_r^+ \to \mathbb{R}$ is a continuous (even Lipschitz) cocycle and $h \in L^p(X_r^+)$. Moreover, $F_t = h \circ T_t - h$ so F is an L^p coboundary. The Livšic regularity theorem for hyperbolic flows [17,27] guarantees that h has a Hölder continuous version.

Now suppose that y is a periodic point of period T and compute that

$$\int_0^T \phi(T_s y) ds = \int_0^1 \left(\int_0^T \phi(T_{s+u} y) ds\right) du = F_T(y) = 0,$$

proving (b).

Finally, it is immediate from Theorem 2 and Remark 4 that $\sigma^2 = 0$ implies that $\widetilde{\sigma}^2 = 0$. □

Remark 7. Ratner [22] proved the CLT for hyperbolic flows and showed that $\widetilde{\sigma}^2 = 0$ if and only if ϕ is an L^2-coboundary (in some sense). However, verifiable criteria for nondegeneracy were first given by [18] who proved the equivalence of (a) and (d) (without requiring rapid mixing).

3. Almost sure invariance principle for hyperbolic flows

In this section, we prove Theorem 1. The proof consists of three ingredients:

(a) Reduction to a suspended flow over a two-sided subshift of finite type, using the symbolic dynamics of Bowen [4,5].
(b) Reduction to the situation where the roof function defining the suspension and the observation ϕ depend only on future coordinates (following [25,6]).
(c) Application of the martingale approximation of Section 2 and standard techniques from probability theory (cf. Conze and le Borgne [8] and Field *et al.* [12]).

3.1. Reduction to a suspended subshift. This step is by now completely standard [4,5,7] and we omit the details. After the reduction, we have a flow on the suspension X_r of an aperiodic two-sided subshift of finite type $\sigma : X \to X$. Here, the roof function $r \in F_\theta(X)$ is strictly positive and the suspension is defined to be $X_r = \{(x,s) \in X \times \mathbb{R} : 0 \le s \le r(x)\}/\sim$ where $(x, r(x)) \sim (\sigma x, 0)$. The suspension flow $T_t(x,s) = (x, s+t)$ is weak mixing with respect to an equilibrium measure $\mu_r = \mu \times \ell / \int r\, d\mu$ where μ is an equilibrium measure on X corresponding to a Hölder potential. The reduced observation ϕ lies in $F_{k,\theta}(X_r)$ and has mean zero. (The spaces $F_\theta(X)$ and $F_{k,\theta}(X_r)$ for the two-sided shift are defined analogously to the one-sided case.)

3.2. Reduction to future coordinates. By [25,6], r is cohomologous to a roof function $r' \in F_{\theta^{1/2}}(X)$ that depends only on future coordinates, and the suspension flows on X_r and $X_{r'}$ are topologically conjugate. Unfortunately, r' is not strictly positive which introduces a number of technical difficulties. (In particular, it is not clear how to define $F_\theta(X_{r'})$.) To circumvent these difficulties, define $r_n = \sum_{j=0}^{n-1} r \circ \sigma^j$. There exists an integer $m \ge 1$ such that r'_m is strictly positive, and it is possible to pass from observations in $F_{k,\theta}(X_r)$ to observations in $F_{k,\theta}(X_{r_m})$ and then to $F_{k,\theta^{1/2}}(X_{r'_m})$ (cf. [10,21]). We omit the tedious details.

The upshot of the discussion above is that without loss of generality we may suppose from the outset that $r \in F_\theta(X)$ depends only on future coordinates. Suppose that $\phi \in F_{k+1,\theta}(X_r)$. A generalization of the argument of [25,6] shows that there is a constant q (depending only on X_r and θ) such that ϕ is cohomologous in $F_{k,\theta^{1/q}}(X_r)$ to an element $\psi \in F_{k,\theta^{1/q}}(X_r)$ depending only on future coordinates. Since we could not find this fact mentioned even implicitly in the literature, we give the proof in detail in the appendix (Theorem 4).

This completes Step (b), and we may suppose without loss that r and ϕ depend only on future coordinates.

3.3. Martingale approximation. This step is almost identical to that in [12] and we only sketch the details. Since the class of hyperbolic sets for smooth flows is closed under time-reversal, it is sufficient to prove the ASIP in reverse time. Hence we consider reverse partial sums $\phi_{-N} = \sum_{j=0}^{N-1} \phi \circ T_{-j}$.

By Lemma 1 (with $n > 4$) and Dolgopyat's results (4), $\phi = \psi + \chi - \chi \circ T_1$ where $\psi, \chi \in L^4$, ψ depends only on future coordinates, and $U^*\psi = 0$. Here, U^*

is the adjoint of the (noninvertible) isometry $U : L^2(X_r^+) \to L^2(X_r^+)$ induced by T_1. As in Remark 3, $\phi_{-N} = \psi_{-N} + o(N^{1/4})$, hence it suffices to prove the ASIP for ψ.

Since ψ and T_1 depend only on future coordinates, the condition $U^*\psi = 0$ guarantees that the sequence $\{\psi_{-N},\ N \in \mathbb{Z}\}$ is a martingale (with respect to the sequence of σ-algebras $T_N(\mathcal{M}^+)$ where $\mathcal{M}^+$ is the σ-algebra on X_r^+ lifted up to X_r). We now apply the method of Strassen [26]. The version stated in [12, Theorem B.3] is sufficient for our purposes. (Hypothesis (a) in [12] is automatically valid since ψ lies in L^4 and the sequence $\psi \circ T_{-j}$ is stationary. Hypothesis (b) follows as in [12] from the strong law of large numbers for martingales since the partial sums of squares also admit a martingale approximation.)

Appendix A. Reduction to future coordinates

Suppose that $\sigma : X \to X$ is a two-sided subshift of finite type. Let $\theta \in (0,1)$, $r \in F_\theta(X)$, and define the suspension X_r corresponding to the roof function r with suspension flow T_t. As described earlier, we define the 'metric' $d_\theta((x,s),(y,t)) = d_\theta(x,y) + |s-t|$. Let $F_\theta(X_r)$ denote the space of *continuous* function $v : X_r \to \mathbb{R}$ that are Lipschitz with respect to the metric d_θ and let $|v|_\theta$ denote the Lipschitz constant.

Remark 8. We have used d_θ to denote the metrics on X and X_r but the context should avoid any ambiguity. Also, it should be noted that d_θ is not really a metric on X_r due to the identifications, but this turns out only to be a minor inconvenience. In this regard, we caution that the continuity assumption for elements of $F_\theta(X_r)$ is not implied by the Lipschitz assumption.

Let $(x,s) \in X_r$. Then $T_t(x,s) = (\sigma^j x, s+t-r_j(x))$, where $s+t \in [r_j(x), r_{j+1}(x))$. The lap number j is a function of x, s, t. Note that $j \in (t/\max r, t/\min r]$.

Proposition 3. *Suppose $x, x' \in X$ and $x_i = x_i'$ for all $i \geq 0$. Then the limit*

$$\Delta(x,x') = \sum_{j=0}^{\infty} \big(r(\sigma^j x) - r(\sigma^j x')\big) = \lim_{j\to\infty} \big(r_j(x) - r_j(x')\big)$$

exists. Moreover, there exists a $t_0 \geq 1$ such that if $x_i = x_i'$ for all $i \geq 0$ and if j and k are the lap numbers corresponding to $T_t(x,s)$ and $T_t(x', s - \Delta(x,x'))$, then $|j-k| \leq 1$ for all $t \geq t_0$.

Proof. Note that $|r(\sigma^j x) - r(\sigma^j x')| \leq |r|_\theta d_\theta(\sigma^j x, \sigma^j x') \leq \theta^j |r|_\theta$ so that Δ is well-defined.

Let j and k be the lap numbers for $T_t(x,s)$ and $T_t(x', s-\Delta(x,x'))$ respectively. Thus $s+t \in [r_j(x), r_{j+1}(x))$ and $s+t \in [r_k(x') - \Delta(x,x'), r_{k+1}(x') - \Delta(x,x'))$.

As $k \to \infty$, the interval $[r_k(x') - \Delta(x,x'), r_{k+1}(x') - \Delta(x,x'))$ converges to the interval $[r_k(x), r_{k+1}(x))$. Hence, within an arbitrarily small error, the intervals $[r_j(x), r_{j+1}(x))$ and $[r_k(x), r_{k+1}(x))$ must eventually overlap. But if $|j-k| \geq 2$, then these intervals are separated by at least distance $\min r$. It follows that eventually $|j-k| \leq 1$. □

Corollary 1. *There exists $N \ge 1$ such that*

$$|v \circ T_n(x,s) - v \circ T_n(x', s - \Delta(x,x'))| \le |v|_\theta \big[1 + |r|_\theta/(1-\theta)\big]\theta^{n/|r|_\infty},$$

for all $v \in F_\theta(X_r)$ and $n \ge N$.

Proof. Denote the lap numbers of $T_n(x,s)$ and $T_n(x', s-\Delta(x,x'))$ by j and k respectively. It follows from Proposition 3 that for each $n \ge N$ large enough, $|j-k| \le 1$. In the case $k = j$,

$$\begin{aligned}|v \circ T_n(x,s) - v \circ T_n(x', s - \Delta(x,x'))| &\le |v|_\theta \big[d_\theta(\sigma^j x, \sigma^j x') + |\Delta(x,x') - r_j(x) + r_j(x')|\big] \\ &\le |v|_\theta \theta^j [1 + |r|_\theta/(1-\theta)] \le |v|_\theta \theta^{n/|r|_\infty}[1 + |r|_\theta/(1-\theta)].\end{aligned}$$

In the case $k = j+1$, we have the estimate

$$\begin{aligned}&\big|v(\sigma^j x, s + n - r_j(x)) - v(\sigma^j x, r(\sigma^j x))\big| + \big|v(\sigma^{j+1}x, 0) - v(\sigma^{j+1}x', s - \Delta(x,x') + n - r_{j+1}(x'))\big| \\ &\le |v|_\theta \big[(r(\sigma^j(x)) - (s + n - r_j(x)) + d_\theta(\sigma^{j+1}x, \sigma^{j+1}x') + (s - \Delta(x,x') + n - r_{j+1}(x')|\big] \\ &= |v|_\theta \big[d_\theta(\sigma^{j+1}x, \sigma^{j+1}x') + (r_{j+1}(x) - r_{j+1}(x') - \Delta(x,x'))|\big] \\ &\le |v|_\theta \theta^{j+1}[1 + |r|_\theta/(1-\theta)] \le |v|_\theta \theta^{n/|r|_\infty}[1 + |r|_\theta/(1-\theta)].\end{aligned}$$

The case $k = j-1$ is similar. $\square$

Proposition 4. *Suppose that $x, x', y, y' \in X$ and $x_i = x_i'$ for $i \ge 0$ and $y_i = y_i'$ for $i \ge 0$. If $d_\theta(x,y) < \theta^{2N}$, $d_\theta(x',y') < \theta^{2N}$ then $|\Delta(x,x') - \Delta(y,y')| < 4|r|_\theta \theta^N/(1-\theta)$.*

Proof. Write

$$\Delta(x,x') - \Delta(y,y') = (r_N(x) - r_N(y)) - (r_N(x') - r_N(y')) + \Delta(\sigma^N x, \sigma^N x') - \Delta(\sigma^N y, \sigma^N y').$$

Now,

$$\begin{aligned}|r_N(x) - r_N(y)| &\le \sum_{j=0}^{N-1} |r(\sigma^j x) - r(\sigma^j y)| \le \sum_{j=0}^{N-1} |r|_\theta d_\theta(\sigma^j x, \sigma^j y) \\ &\le \sum_{j=0}^{N-1} |r|_\theta \theta^{-j} d_\theta(x,y) \le |r|_\theta \theta^{-N} d_\theta(x,y)/(1-\theta) \le |r|_\theta \theta^N/(1-\theta),\end{aligned}$$

and similarly for $r_N(x') - r_N(y')$. Next, compute that

$$|\Delta(\sigma^N x, \sigma^N x')| \le \sum_{j=N}^{\infty} |r(\sigma^j x) - r(\sigma^j x')| \le |r|_\theta \sum_{j=N}^{\infty} \theta^j = |r|_\theta \theta^N/(1-\theta),$$

and similarly for $\Delta(\sigma^N y, \sigma^N y')$. $\square$

Let $\partial_t v = (\partial/\partial_t)(v \circ T_t)|_{t=0}$ denote the derivative of $v : X_r \to \mathbb{R}$ in the flow direction. Let $F_{k,\theta}(X_r)$ denote the space of functions $v : X_r \to \mathbb{R}$ such that $\partial_t^j v \in F_\theta(X_r)$ for $j = 0, \ldots, k$ and define $|v|_{k,\theta} = \max_{j=0,\ldots,k} |\partial_t^j v|_\theta$.

Theorem 4. *Let $\sigma : X \to X$ be a two-sided subshift and let $r \in F_\theta(X)$ be a roof function, $r > 0$. Suppose further that r depends only on future coordinates. Define $q = (4 + 2|1/r|_\infty)|r|_\infty$.*

Let $v \in F_{k+1,\theta}(X_r)$. Then there exists $w, \chi \in F_{k,\theta^{1/q}}(X_r)$ such that w depends only on future coordinates, and $v = w + \chi - \chi \circ T_1$.

Proof. For each letter a, choose an element $x^a \in X$ such that $(x^a)_0 = a$. Given $x \in X$ define $\varphi(x) \in X$ as follows: $(\varphi(x))_i = x_i$ for $i \geq 0$ and $(\varphi(x))_i = (x^{x_0})_i$ for $i \leq 0$. So the future coordinates of $\varphi(x)$ agree with x whereas the past coordinates of $\varphi(x)$ depend only on x_0. In particular, the map $\varphi : X \to X$ depends only on future coordinates.

By Proposition 3, we can define $\widetilde{\varphi}(x, s) = (\varphi x, s - \Delta(x, \varphi x))$. Define (formally for the moment)

$$\chi = \sum_{n=0}^{\infty} (v \circ T_n - v \circ T_n \circ \widetilde{\varphi}).$$

Compute that $v = w + \chi - \chi \circ T_1$ where $w = \sum_{n=0}^{\infty} (v \circ T_n \circ \widetilde{\varphi} - v \circ T_n \circ \widetilde{\varphi} \circ T_1)$, which clearly depends only on future coordinates (since φ and r (hence T_t, $t > 0$) depend only on future coordinates). It remains to show that χ (and hence w) lies in $F_{k,\theta^{1/q}}(X_r)$.

First, we show that χ is C^{k+1} in the flow direction. Differentiating χ formally term by term yields the series $\partial_t^j \chi = \sum_{n=0}^{\infty} ((\partial_t^j v) \circ T_n - (\partial_t^j v) \circ T_n \circ \widetilde{\varphi})$. For fixed $0 \leq j \leq k+1$, since $\partial_t^j v \in F_\theta(X_r)$, we deduce from Proposition 3 that the n'th term of $\partial_t \chi$ is bounded in absolute value by $|\partial_t^j v|_\theta \theta^{n/|r|_\infty} [1 + |r|_\theta/(1-\theta)]$ and so the series converges uniformly to a continuous function $\partial_t^j \chi$. In particular, χ is C^{k+1} in the flow direction.

It remains to show that $\partial_t^j \chi$ is Lipschitz with respect to the $d_{\theta^{1/q}}$ metric for all $0 \leq j \leq k$. It suffices to show that χ is Lipschitz with respect to the $d_{\theta^{1/q}}$ metric under the assumption that $v \in F_{1,\theta}$ (the general case follows replacing v by $\partial_t^j v$). Moreover, since χ is C^1 and hence Lipschitz in the flow direction (which we can identify with the s variable), we may keep the s variable fixed.

Choose N large as in Proposition 3. In analogy with the proof of Proposition 4, we have the decomposition $|\chi(x,s) - \chi(y,s)| \leq A_1(x,y) + A_2(x,y) + B(x) + B(y)$, where

$$\begin{aligned}
A_1(x,y) &= \sum_{n=0}^{N} |v \circ T_n(x,s) - v \circ T_n(y,s)|, \\
A_2(x,y) &= \sum_{n=0}^{N} |v \circ T_n(\widetilde{\varphi}(x,s)) - v \circ T_n(\widetilde{\varphi}(y,s))|, \\
B(x) &= \sum_{n=N+1}^{\infty} |v \circ T_n(x,s) - v \circ T_n(\widetilde{\varphi}(x,s))|.
\end{aligned}$$

Let $q_1 = |r|_\infty$ and $q_2 = 2 + |1/r|_\infty$. We claim that provided N is large enough (independent of v), there exists a constant $K > 0$ such that (i) $B(x) \leq K\theta^{N/q_1}$ for all $x \in X$, and (ii) $A_1(x,y), A_2(x,y) \leq K\theta^{N/2}$ for all $x, y \in X$ with $d_\theta(x,y) < \theta^{Nq_2}$. Let $q = 2q_1q_2$. It then follows that $|\chi(x,s) - \chi(y,s)| \leq 4K d_{\theta^{1/q}}(x,y)$ proving the result.

As before, the n'th term of $B(x)$ is dominated by $C\theta^{n/|r|_\infty} = C\theta^{n/q_1}$, verifying (i). It remains to verify (ii). We give the details for the more difficult term $A_2(x,y)$.

Choose N so large that $4|r|_\theta\theta^N/(1-\theta) < \min r/2$ and $N\theta^{N/2} < 1$.

Suppose that $d_\theta(x,y) < \theta^{Nq_2}$. By Proposition 4, $|\Delta(x,\varphi x) - \Delta(y,\varphi y)| < \min r/2$. Also,

$$\begin{aligned}|r_j(\varphi x) - r_j(\varphi y)| &\le |r|_\theta\theta^{-j+1}\theta^{Nq_2}/(1-\theta) \le |r|_\theta\theta^{N(q_2-|1/r|_\infty)}/(1-\theta)\\ &= |r|_\theta\theta^{2N}/(1-\theta) < \min r/2,\end{aligned}$$

for all $1 \le j \le [N|1/r|_\infty] + 1$. Hence for this range of j, the intervals $[r_j(\varphi x) + \Delta(x,\varphi x), r_{j+1}(\varphi x) + \Delta(x,\varphi x))$ and $[r_j(\varphi y) + \Delta(y,\varphi y), r_{j+1}(\varphi y) + \Delta(y,\varphi y))$ almost coincide (the initial points are within distance $\min r$, as are the final points). It follows as in the proof of Proposition 3 that the lap numbers j and k of $T_n(\widetilde{\varphi}(x,s))$ and $T_n(\widetilde{\varphi}(y,s))$ satisfy $|j-k| \le 1$ for all $0 \le n \le N$. The estimation of the terms in $A_2(x,y)$ now splits into three cases as in the proof of Corollary 1. When $j = k$, we obtain the term

$$v\big(\sigma^j\varphi x, s - \Delta(x,\varphi x) + n - r_j(\varphi x)\big) - v\big(\sigma^j\varphi y, s - \Delta(y,\varphi y) + n - r_j(\varphi y)\big),$$

which is dominated by

$$\begin{aligned}&|v|_\theta\Big\{d_\theta(\sigma^j\varphi x, \sigma^j\varphi y) + |r_j(\varphi x) - r_j(\varphi y)| + |\Delta(x,\varphi x) - \Delta(y,\varphi y)|\Big\}\\ &\quad\le |v|_\theta\Big\{[1 + |r|_\theta/(1-\theta)]\theta^{-j}d_\theta(\varphi x,\varphi y) + 4|r|_\theta\theta^N/(1-\theta)\Big\}\\ &\quad\le |v|_\theta\Big\{[1 + |r|_\theta/(1-\theta)]\theta^{Nq_2-n|1/r|_\infty} + 4|r|_\theta\theta^N/(1-\theta)\Big\}.\end{aligned}$$

The computations for $j = k \pm 1$ lead to the same estimates (just as in the proof of Corollary 1) and summing the terms we obtain

$$\begin{aligned}A_2(x,y) &\le |v|_\theta\Big\{[1 + |r|_\theta/(1-\theta)]\theta^{N(q_2-|1/r|_\infty)}/(1-\theta) + 4|r|_\theta N\theta^N/(1-\theta)\Big\}\\ &\le |v|_\theta\Big\{[1 + |r|_\theta/(1-\theta)]\theta^{2N}/(1-\theta) + 4|r|_\theta\theta^{N/2}/(1-\theta)\Big\},\end{aligned}$$

(since $N\theta^{N/2} < 1$) completing the proof. □

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