$$= \oint_{C_2} \frac{\partial}{\partial z'}((x'-x)^2 + (y'-y)^2 + (z'-z)^2)^{-3/2} dz' = 0,$$

where the last equality is a consequence of the fundamental theorem of calculus. Of the two, only $\frac{\partial Q}{\partial x}$ has a dx' in it, and that part is

$$3\oint_{C_2} ((x-x')^2 + (y-y')^2 + (z-z')^2)^{-5/2}(x-x')(z-z')dx'$$
$$= \oint_{C_2} \frac{\partial}{\partial x'} \frac{z-z'}{((x-x')^2 + (y-y')^2 + (z-z')^2)^{3/2}} dx' = 0.$$

The term involving dy' is treated similarly. The conclusion follows.

Remark. The linking number is, in fact, an integer, which measures the number of times the curves wind around each other. It was defined by C.F. Gauss, who used it to decide based on astronomical observations, whether the orbits of certain asteroids were winding around the orbit of the earth.

535. Plugging in $x = y$, we find that $f(0) = 0$, and plugging in $x = -1$, $y = 0$, we find that $f(1) = -f(-1)$. Also, plugging in $x = a$, $y = 1$, and then $x = a$, $y = -1$, we obtain

$$f(a^2 - 1) = (a-1)(f(a) + f(1)),$$
$$f(a^2 - 1) = (a+1)(f(a) - f(1)).$$

Equating the right-hand sides and solving for $f(a)$ gives $f(a) = f(1)a$ for all a.

So any such function is linear. Conversely, a function of the form $f(x) = kx$ clearly satisfies the equation.

(Korean Mathematical Olympiad, 2000)

536. Replace z by $1 - z$ to obtain

$$f(1-z) + (1-z)f(z) = 2 - z.$$

Combine this with $f(z) + zf(1-z) = 1 + z$, and eliminate $f(1-z)$ to obtain

$$(1 - z + z^2)f(z) = 1 - z + z^2.$$

Hence $f(z) = 1$ for all z except maybe for $z = e^{\pm\pi i/3}$, when $1 - z + z^2 = 0$. For $\alpha = e^{i\pi/3}$, $\bar{\alpha} = \alpha^2 = 1 - \alpha$; hence $f(\alpha) + \alpha f(\bar{\alpha}) = 1 + \alpha$. We therefore have only one constraint, namely $f(\bar{\alpha}) = [1 + \alpha - f(\alpha)]/\alpha = \bar{\alpha} + 1 - \bar{\alpha} f(\alpha)$. Hence the solution to the functional equation is of the form

$$f(z) = 1 \quad \text{for } z \neq e^{\pm i\pi/3},$$

$$f(e^{i\pi/3}) = \beta,$$
$$f(e^{-i\pi/3}) = \bar{\alpha} + 1 - \bar{\alpha}\beta,$$

where β is an arbitrary complex parameter.
(20th W.L. Putnam Competition, 1959)

537. Successively, we obtain

$$f(-1) = f\left(-\frac{1}{2}\right) = f\left(-\frac{1}{3}\right) = \cdots = \lim_{n\to\infty} f\left(-\frac{1}{n}\right) = f(0).$$

Hence $f(x) = f(0)$ for $x \in \{0, -1, -\frac{1}{2}, \ldots, -\frac{1}{n}, \ldots\}$.
If $x \neq 0, -1, \ldots, -\frac{1}{n}, \ldots$, replacing x by $\frac{x}{1+x}$ in the functional equation, we obtain

$$f\left(\frac{x}{1+x}\right) = f\left(\frac{\frac{x}{1+x}}{1 - \frac{x}{1+x}}\right) = f(x).$$

And this can be iterated to yield

$$f\left(\frac{x}{1+nx}\right) = f(x), \quad n = 1, 2, 3 \ldots.$$

Because f is continuous at 0 it follows that

$$f(x) = \lim_{n\to\infty} f\left(\frac{x}{1+nx}\right) = f(0).$$

This shows that only constant functions satisfy the functional equation.

538. Plugging in $x = t$, $y = 0$, $z = 0$ gives

$$f(t) + f(0) + f(t) \geq 3f(t),$$

or $f(0) \geq f(t)$ for all real numbers t. Plugging in $x = \frac{t}{2}$, $y = \frac{t}{2}$, $z = -\frac{t}{2}$ gives

$$f(t) + f(0) + f(0) \geq 3f(0),$$

or $f(t) \geq f(0)$ for all real numbers t. Hence $f(t) = f(0)$ for all t, so f must be constant. Conversely, any constant function f clearly satisfies the given condition.
(Russian Mathematical Olympiad, 2000)

539. No! In fact, we will prove a more general result.

Proposition. *Let S be a set and $g : S \to S$ a function that has exactly two fixed points $\{a, b\}$ and such that $g \circ g$ has exactly four fixed points $\{a, b, c, d\}$. Then there is no function $f : S \to S$ such that $g = f \circ f$.*

Proof. Let $g(c) = y$. Then $c = g(g(c)) = g(y)$; hence $y = g(c) = g(g(y))$. Thus y i a fixed point of $g \circ g$. If $y = a$, then $a = g(a) = g(y) = c$, leading to a contradiction Similarly, $y = b$ forces $c = b$. If $y = c$, then $c = g(y) = g(c)$, so c is a fixed point o g, again a contradiction. It follows that $y = d$, i.e., $g(c) = d$, and similarly $g(d) = c$.

Suppose there is $f : S \to S$ such that $f \circ f = g$. Then $f \circ g = f \circ f \circ f = g \circ f$ Then $f(a) = f(g(a)) = g(f(a))$, so $f(a)$ is a fixed point of g. Examining case by case, we conclude that $f(\{a, b\}) \subset \{a, b\}$ and $f(\{a, b, c, d\}) \subset \{a, b, c, d\}$. Because $f \circ f = g$, the inclusions are, in fact, equalities.

Consider $f(c)$. If $f(c) = a$, then $f(a) = f(f(c)) = g(c) = d$, a contradiction since $f(a)$ is in $\{a, b\}$. Similarly, we rule out $f(c) = b$. Of course, c is not a fixed poin of f, since it is not a fixed point of g. We are left with the only possibility $f(c) = d$ But then $f(d) = f(f(c)) = g(c) = d$, and this again cannot happen because d is not a fixed point of g. We conclude that such a function f cannot exist.

In the particular case of our problem, $g(x) = x^2 - 2$ has the fixed points -1 and 2, and $g(g(x)) = (x^2 - 2)^2 - 2$ has the fixed points $-1, 2, \frac{-1+\sqrt{5}}{2}$, and $\frac{-1-\sqrt{5}}{2}$. This completes the solution.

(B.J. Venkatachala, *Functional Equations: A Problem Solving Approach*, Prism Books PVT Ltd., 2002)

540. The standard approach is to substitute particular values for x and y. The solution found by the student S.P. Tungare does quite the opposite. It introduces an additional variable z. The solution proceeds as follows:

$$\begin{aligned}
&f(x + y + z)\\
&\quad = f(x)f(y + z) - c \sin x \sin(y + z)\\
&\quad = f(x)[f(y)f(z) - c \sin y \sin z] - c \sin x \sin y \cos z - c \sin x \cos y \sin z\\
&\quad = f(x)f(y)f(z) - cf(x) \sin y \sin z - c \sin x \sin y \cos z - c \sin x \cos y \sin z.
\end{aligned}$$

Because obviously $f(x + y + z) = f(y + x + z)$, it follows that we must have

$$\sin z[f(x) \sin y - f(y) \sin x] = \sin z[\cos x \sin y - \cos y \sin x].$$

Substitute $z = \frac{\pi}{2}$ to obtain

$$f(x) \sin y - f(y) \sin x = \cos x \sin y - \cos y \sin x.$$

For $x = \pi$ and y not an integer multiple of π, we obtain $\sin y[f(\pi) + 1] = 0$, and hence $f(\pi) = -1$.

Then, substituting in the original equation $x = y = \frac{\pi}{2}$ yields

$$f(\pi) = \left[f\left(\frac{\pi}{2}\right)\right] - c,$$

whence $f(\frac{\pi}{2}) = \pm\sqrt{c-1}$. Substituting in the original equation $y = \pi$ we also obtain $f(x+\pi) = -f(x)$. We then have

$$\begin{aligned}-f(x) = f(x+\pi) &= f\left(x+\frac{\pi}{2}\right)f\left(\frac{\pi}{2}\right) - c\cos x\\ &= f\left(\frac{\pi}{2}\right)\left(f(x)f\left(\frac{\pi}{2}\right) - c\sin x\right) - c\cos x,\end{aligned}$$

whence

$$f(x)\left[\left(f\left(\frac{\pi}{2}\right)\right)^2 - 1\right] = cf\left(\frac{\pi}{2}\right)\sin x - c\cos x.$$

It follows that $f(x) = f(\frac{\pi}{2})\sin x + \cos x$. We find that the functional equation has two solutions, namely,

$$f(x) = \sqrt{c-1}\sin x + \cos x \quad \text{and} \quad f(x) = -\sqrt{c-1}\sin x + \cos x.$$

(Indian Team Selection Test for the International Mathematical Olympiad, 2004)

541. Because $|f|$ is bounded and is identically equal to zero, its supremum is a positive number M. Using the equation from the statement and the triangle inequality, we obtain that for any x and y,

$$\begin{aligned}2|f(x)||g(y)| &= |f(x+y) + f(x-y)|\\ &\le |f(x+y)| + |f(x-y)| \le 2M.\end{aligned}$$

Hence

$$|g(y)| \le \frac{M}{|f(x)|}.$$

If in the fraction on the right we take the supremum of the denominator, we obtain $|g(y)| \le \frac{M}{M} = 1$ for all y, as desired.

Remark. The functions $f(x) = \sin x$ and $g(x) = \cos x$ are an example.

(14th International Mathematical Olympiad, 1972)

542. Substituting for f a linear function $ax + b$ and using the method of undetermined coefficients, we obtain $a = 1, b = -\frac{3}{2}$, so $f(x) = x - \frac{3}{2}$ is a solution.

Are there other solutions? Setting $g(x) = f(x) - (x - \frac{3}{2})$, we obtain the simpler functional equation

$$3g(2x+1) = g(x), \quad \text{for all } x \in \mathbb{R}.$$

This can be rewritten as

$$g(x) = \frac{1}{3} g\left(\frac{x-1}{2}\right), \quad \text{for all } x \in \mathbb{R}.$$

For $x = -1$ we have $g(-1) = \frac{1}{3}g(-1)$; hence $g(-1) = 0$. In general, for an arbitrary x, define the recursive sequence $x_0 = x$, $x_{n+1} = \frac{x_n - 1}{2}$ for $n \geq 0$. It is not hard to see that this sequence is Cauchy, for example, because $|x_{m+n} - x_m| \leq \frac{1}{2^{m-2}} \max(1, |x|)$. This sequence is therefore convergent, and its limit L satisfies the equation $L = \frac{L-1}{2}$. It follows that $L = -1$. Using the functional equation, we obtain

$$g(x) = \frac{1}{3}g(x_1) = \frac{1}{9}g(x_2) = \cdots = \frac{1}{3^n}g(x_n).$$

Passing to the limit, we obtain $g(x) = 0$. This shows that $f(x) = x - \frac{3}{2}$ is the unique solution to the functional equation.

(B.J. Venkatachala, *Functional Equations: A Problem Solving Approach*, Prism Books PVT Ltd., 2002)

543. We will first show that $f(x) \geq x$ for all x. From (i) we deduce that $f(3x) \geq 2x$, so $f(x) \geq \frac{2x}{3}$. Also, note that if there exists k such that $f(x) \geq kx$ for all x, then $f(x) \geq \frac{k^3+2}{3}x$ for all x as well. We can iterate and obtain $f(x) \geq k_n x$, where k_n are the terms of the recursive sequence defined by $k_1 = \frac{2}{3}$, and $k_{n+1} = \frac{k_n^3+2}{3}$ for $k \geq 1$. Let us examine this sequence.

By the AM–GM inequality,

$$k_{n+1} = \frac{k_n^3 + 1^3 + 1^3}{3} \geq k_n,$$

so the sequence is increasing. Inductively we prove that $k_n < 1$. Weierstrass' criterion implies that $(k_n)_n$ is convergent. Its limit L should satisfy the equation

$$L = \frac{L^3 + 2}{3},$$

which shows that L is a root of the polynomial equation $L^3 - 3L + 2 = 0$. This equation has only one root in $[0, 1]$, namely $L = 1$. Hence $\lim_{n\to\infty} k_n = 1$, and so $f(x) \geq x$ for all x.

It follows immediately that $f(3x) \geq 2x + f(x)$ for all x. Iterating, we obtain that for all $n \geq 1$,

$$f(3^n x) - f(x) \geq (3^n - 1)x.$$

Therefore, $f(x) - x \leq f(3^n x) - 3^n x$. If we let $n \to \infty$ and use (ii), we obtain $f(x) - x \leq 0$, that is, $f(x) \leq x$. We conclude that $f(x) = x$ for all $x > 0$. Thus the identity function is the unique solution to the functional equation.

(G. Dospinescu)

544. We should keep in mind that $f(x) = \sin x$ and $g(x) = \cos x$ satisfy the condition. As we proceed with the solution to the problem, we try to recover some properties of $\sin x$ and $\cos x$. First, note that the condition $f(t) = 1$ and $g(t) = 0$ for some $t \neq 0$ implies $g(0) = 1$; hence g is nonconstant. Also, $0 = g(t) = g(0)g(t) + f(0)f(t) = f(0)$; hence f is nonconstant. Substituting $x = 0$ in the relation yields $g(-y) = g(y)$, so g is even.

Substituting $y = t$, we obtain $g(x - t) = f(x)$, with its shifted version $f(x + t) = g(x)$. Since g is even, it follows that $f(-x) = g(x + t)$. Now let us combine these facts to obtain

$$\begin{aligned} f(x - y) &= g(x - y - t) = g(x)g(y + t) + f(x)f(y + t) \\ &= g(x)f(-y) + f(x)g(y). \end{aligned}$$

Change y to $-y$ to obtain $f(x + y) = f(x)g(y) + g(x)f(y)$ (the addition formula for sine).

The remaining two identities are consequences of this and the fact that f is odd. Let us prove this fact. From $g(x - (-y)) = g(x + y) = g(-x - y)$, we obtain

$$f(x)f(-y) = f(y)f(-x)$$

for all x and y in $\mathbb{R}$. Setting $y = t$ and $x = -t$ yields $f(-t)^2 = 1$, so $f(-t) = \pm 1$. The choice $f(-t) = 1$ gives $f(x) = f(x)f(-t) = f(-x)f(t) = f(-x)$; hence f is even. But then

$$f(x - y) = f(x)g(-y) + g(x)f(-y) = f(x)g(y) + g(x)f(y) = f(x + y),$$

for all x and y. For $x = \frac{z+w}{2}$, $y = \frac{z-w}{2}$, we have $f(z) = f(w)$, and so f is constant, a contradiction. For $f(-t) = -1$, we obtain $f(-x) = -f(-x)f(-t) = -f(x)f(t) = -f(x)$; hence f is odd. It is now straightforward that

$$f(x - y) = f(x)g(y) + g(x)f(-y) = f(x)g(y) - g(x)f(y)$$

and

$$g(x + y) = g(x - (-y)) = g(x)g(-y) + f(x)f(-y) = g(x)g(y) - f(x)f(y),$$

where in the last equality we also used the fact, proved above, that g is even.

(*American Mathematical Monthly*, proposed by V.L. Klee, solution by P.L. Kannappan)

545. Because $f(x) = f^2(x/2) > 0$, the function $g(x) = \ln f(x)$ is well defined. It satisfies Cauchy's equation and is continuous; therefore, $g(x) = \alpha x$ for some constant α. We obtain $f(x) = c^x$, with $c = e^{\alpha}$.

546. Adding 1 to both sides of the functional equation and factoring, we obtain

$$f(x+y)+1=(f(x)+1)(f(y)+1).$$

The continuous function $g(x)=f(x)+1$ satisfies the functional equation $g(x+y)=g(x)g(y)$, and we have seen in the previous problem that $g(x)=c^x$ for some nonnegativ constant c. We conclude that $f(x)=c^x-1$ for all x.

547. If there exists x_0 such that $f(x_0)=1$, then

$$f(x)=f(x_0+(x-x_0))=\frac{1+f(x-x_0)}{1+f(x-x_0)}=1.$$

In this case, f is identically equal to 1. In a similar manner, we obtain the constan solution $f(x)\equiv -1$.

Let us now assume that f is never equal to 1 or -1. Define $g:\mathbb{R}\to\mathbb{R}$, $g(x)=\frac{1+f(x)}{1-f(x)}$. To show that g is continuous, note that for all x,

$$f(x)=\frac{2f\left(\frac{x}{2}\right)}{1+f\left(\frac{x}{2}\right)}<1.$$

Now the continuity of g follows from that of f and of the function $h(t)=\frac{1+t}{1-t}$ on $(-\infty,1)$. Also,

$$\begin{aligned}g(x+y)&=\frac{1+f(x+y)}{1-f(x+y)}=\frac{f(x)f(y)+1+f(x)+f(y)}{f(x)f(y)+1-f(x)-f(y)}\\&=\frac{1+f(x)}{1-f(x)}\cdot\frac{1+f(y)}{1-f(y)}=g(x)g(y).\end{aligned}$$

Hence g satisfies the functional equation $g(x+y)=g(x)g(y)$. As seen in problem 545, $g(x)=c^x$ for some $c>0$. We obtain $f(x)=\frac{c^x-1}{c^x+1}$. The solutions to the equation are therefore

$$f(x)=\frac{c^x-1}{c^x+1},\qquad f(x)=1,\qquad f(x)=-1.$$

Remark. You might have recognized the formula for the hyperbolic tangent of the sum. This explains the choice of g, by expressing the exponential in terms of the hyperbolic tangent.

548. Rewrite the functional equation as

$$\frac{f(xy)}{xy}=\frac{f(x)}{x}+\frac{f(y)}{y}.$$

It now becomes natural to let $g(x) = \frac{f(x)}{x}$, which satisfies the equation

$$g(xy) = g(x) + g(y).$$

The particular case $x = y$ yields $g(x) = \frac{1}{2}g(x^2)$, and hence $g(-x) = \frac{1}{2}g((-x)^2) = \frac{1}{2}g(x^2) = g(x)$. Thus we only need to consider the case $x > 0$.

Note that g is continuous on $(0, \infty)$. If we compose g with the continuous function $h : \mathbb{R} \to (0, \infty)$, $h(x) = e^x$, we obtain a continuous function on $\mathbb{R}$ that satisfies Cauchy's equation. Hence $g \circ h$ is linear, which then implies $g(x) = \log_a x$ for some positive base a. It follows that $f(x) = x \log_a x$ for $x > 0$ and $f(x) = x \log_a |x|$ if $x < 0$.

All that is missing is the value of f at 0. This can be computed directly setting $x = y = 0$, and it is seen to be 0. We conclude that $f(x) = x \log_a |x|$ if $x \neq 0$, and $f(0) = 0$, where a is some positive number. The fact that any such function is continuous at zero follows from

$$\lim_{x \to 0+} x \log_a x = 0,$$

which can be proved by applying the L'Hôpital's theorem to the functions $\log_a x$ and $\frac{1}{x}$. This concludes the solution.

549. Setting $y = z = 0$ yields $\phi(x) = f(x) + g(0) + h(0)$, and similarly $\phi(y) = g(y) + f(0) + h(0)$. Substituting these three relations in the original equation and letting $z = 0$ gives rise to a functional equation for ϕ, namely

$$\phi(x + y) = \phi(x) + \phi(y) - (f(0) + g(0) + h(0)).$$

This should remind us of the Cauchy equation, which it becomes after changing the function ϕ to $\psi(x) = \phi(x) - (f(0) + g(0) + h(0))$. The relation $\psi(x + y) = \psi(x) + \psi(y)$ together with the continuity of ψ shows that $\psi(x) = cx$ for some constant c. We obtain the solution to the original equation

$$\phi(x) = cx + \alpha + \beta + \gamma, \qquad f(x) = cx + \alpha, \qquad g(x) = cx + \beta, \qquad h(x) = cx + \gamma,$$

where α, β, γ are arbitrary real numbers.

(*Gazeta Matematică* (*Mathematics Gazette, Bucharest*), proposed by M. Vlada)

550. This is a generalization of Cauchy's equation. Trying small values of n, one can guess that the answer consists of all polynomial functions of degree at most $n - 1$ with no constant term (i.e., with $f(0) = 0$). We prove by induction on n that this is the case.

The case $n = 2$ is Cauchy's equation. Assume that the claim is true for $n - 1$ and let us prove it for n. Fix x_n and consider the function $g_{x_n} : \mathbb{R} \to \mathbb{R}$, $g_{x_n}(x) = f(x + x_n) - f(x) - f(x_n)$. It is continuous. More importantly, it satisfies the functional equation for $n - 1$. Hence $g_{x_n}(x)$ is a polynomial of degree $n - 2$. And this is true for all x_n.

It follows that $f(x + x_n) - f(x)$ is a polynomial of degree $n - 2$ for all x
particular, there exist polynomials $P_1(x)$ and $P_2(x)$ such that $f(x+1) - f(x) =$
and $f(x + \sqrt{2}) - f(x) = P_2(x)$. Note that for any a, the linear map from the
space of polynomials of degree at most $n - 1$ to the vector space of polynomi
degree at most $n - 2$, $P(x) \to P(x + a) - P(x)$, has kernel the one-dimensional
of constant polynomials (the only periodic polynomials). Because the first vector
has dimension n and the second has dimension $n - 1$, the map is onto. Hence there
polynomials $Q_1(x)$ and $Q_2(x)$ of degree at most $n - 1$ such that

$$Q_1(x+1) - Q_1(x) = P_1(x) = f(x+1) - f(x),$$
$$Q_2(x+\sqrt{2}) - Q_2(x) = P_2(x) = f(x+\sqrt{2}) - f(x).$$

We deduce that the functions $f(x) - Q_1(x)$ and $f(x) - Q_2(x)$ are continuous and peri
hence bounded. Their difference $Q_1(x) - Q_2(x)$ is a bounded polynomial, hence cons
Consequently, the function $f(x) - Q_1(x)$ is continuous and has the periods 1 and
Since the additive group generated by 1 and $\sqrt{2}$ is dense in $\mathbb{R}$, $f(x) - Q_1(x)$ is const
This completes the induction.

That any polynomial of degree at most $n - 1$ with no constant term satisfies
functional equation also follows by induction on n. Indeed, the fact that f satisfies
equation is equivalent to the fact that g_{x_n} satisfies the equation. And g_{x_n} is a polynon
of degree $n - 2$.

(G. Dospinescu)

551. *First solution*: Assume that such functions do exist. Because $g \circ f$ is a bijection
is one-to-one and g is onto. Since f is a one-to-one continuous function, it is monoton
and because g is onto but $f \circ g$ is not, it follows that f maps $\mathbb{R}$ onto an interval I stric
included in $\mathbb{R}$. One of the endpoints of this interval is finite, call this endpoint a. Witho
loss of generality, we may assume that $I = (a, \infty)$. Then as $g \circ f$ is onto, $g(I) =$
This can happen only if $\limsup_{x\to\infty} g(x) = \infty$ and $\liminf_{x\to\infty} g(x) = -\infty$, whi
means that g oscillates in a neighborhood of infinity. But this is impossible becau
$f(g(x)) = x^2$ implies that g assumes each value at most twice. Hence the question h
a negative answer; such functions do not exist.

Second solution: Since $g \circ f$ is a bijection, f is one-to-one and g is onto. Note th
$f(g(0)) = 0$. Since g is onto, we can choose a and b with $g(a) = g(0) - 1$ and $g(b)$
$g(0) + 1$. Then $f(g(a)) = a^2 > 0$ and $f(g(b)) = b^2 > 0$. Let $c = \min(a^2, b^2)/2$
0. The intermediate value property guarantees that there is an $x_0 \in (g(a), g(0))$ wit
$f(x_0) = c$ and an $x_1 \in (g(0), g(b))$ with $f(x_1) = c$. This contradicts the fact that f i
one-to-one. Hence no such functions can exist.

(R. Gelca, second solution by R. Stong)

552. The relation from the statement implies that f is injective, so it must be monotonic
Let us show that f is increasing. Assuming the existence of a decreasing solution f t

the functional equation, we can find x_0 such that $f(x_0) \neq x_0$. Rewrite the functional equation as $f(f(x)) - f(x) = f(x) - x$. If $f(x_0) < x_0$, then $f(f(x_0)) < f(x_0)$, and if $f(x_0) > x_0$, then $f(f(x_0)) > f(x_0)$, which both contradict the fact that f is decreasing. Thus any function f that satisfies the given condition is increasing.

Pick some $a > b$, and set $\Delta f(a) = f(a) - a$ and $\Delta f(b) = f(b) - b$. By adding a constant to f (which yields again a solution to the functional equation), we may assume that $\Delta f(a)$ and $\Delta f(b)$ are positive. Composing f with itself n times, we obtain $f^{(n)}(a) = a + n\Delta f(a)$ and $f^{(n)}(b) = b + n\Delta f(b)$. Recall that f is an increasing function, so $f^{(n)}$ is increasing, and hence $f^{(n)}(a) > f^{(n)}(b)$, for all n. This can happen only if $\Delta f(a) \geq \Delta f(b)$.

On the other hand, there exists m such that $b + m\Delta f(b) = f^{(m)}(b) > a$, and the same argument shows that $\Delta f(f^{(m-1)}(b)) > \Delta f(a)$. But $\Delta f(f^{(m-1)}(b)) = \Delta f(b)$, so $\Delta f(b) \geq \Delta f(a)$. We conclude that $\Delta f(a) = \Delta f(b)$, and hence $\Delta f(a) = f(a) - a$ is independent of a. Therefore, $f(x) = x + c$, with $c \in \mathbb{R}$, and clearly any function of this type satisfies the equation from the statement.

553. The answer is yes! We have to prove that for $f(x) = e^{x^2}$, the equation $f'g + fg' = f'g'$ has nontrivial solutions on some interval (a, b). Explicitly, this is the first-order linear equation in g,

$$(1 - 2x)e^{x^2}g' + 2xe^{x^2}g = 0.$$

Separating the variables, we obtain

$$\frac{g'}{g} = \frac{2x}{2x - 1} = 1 + \frac{1}{2x - 1},$$

which yields by integration $\ln g(x) = x + \frac{1}{2}\ln|2x - 1| + C$. We obtain the one-parameter family of solutions

$$g(x) = ae^x\sqrt{|2x - 1|}, \quad a \in \mathbb{R},$$

on any interval that does not contain $\frac{1}{2}$.

(49th W.L. Putnam Mathematical Competition, 1988)

554. Rewrite the equation $f^2 + g^2 = f'^2 + g'^2$ as

$$(f + g)^2 + (f - g)^2 = (f' + g')^2 + (g' - f')^2.$$

This, combined with $f + g = g' - f'$, implies that $(f - g)^2 = (f' + g')^2$.

Let x_0 be the second root of the equation $f(x) = g(x)$. On the intervals $I_1 = (-\infty, 0)$, $I_2 = (0, x_0)$, and $I_3 = (x_0, \infty)$ the function $f - g$ is nonzero; hence so is $f' + g'$. These two functions maintain constant sign on the three intervals; hence $f - g = \epsilon_j(f' + g')$ on I_j, for some $\epsilon_j \in \{-1, 1\}$, $j = 1, 2, 3$.

If on any of these intervals $f - g = f' + g'$, then since $f + g = g' - f'$ it follo that $f = g'$ on that interval, and so $g' + g = g' - g''$. This implies that g satisf the equation $g'' + g = 0$, or that $g(x) = A \sin x + B \cos x$ on that interval. Al $f(x) = g'(x) = A \cos x - B \sin x$.

If $f - g = -f' - g'$ on some interval, then using again $f + g = g' - f'$, we fi that $g = g'$ on that interval. Hence $g(x) = C_1 e^x$. From the fact that $f = -f'$, we obt $f(x) = C_2 e^{-x}$.

Assuming that f and g are exponentials on the interval $(0, x_0)$, we deduce t $C_1 = g(0) = f(0) = C_2$ and that $C_1 e^{x_0} = g(x_0) = f(x_0) = C_2 e^{-x}$. These t inequalities cannot hold simultaneously, unless f and g are identically zero, ruled o by the hypothesis of the problem. Therefore, $f(x) = A \cos x - B \sin x$ and $g(x)$ $A \sin x + B \cos x$ on $(0, x_0)$, and consequently $x_0 = \pi$.

On the intervals $(-\infty, 0]$ and $[x_0, \infty)$ the functions f and g cannot be periodic, sin then the equation $f = g$ would have infinitely many solutions. So on these interv the functions are exponentials. Imposing differentiability at 0 and π, we obtain $B =$ $C_1 = A$ on I_1 and $C_1 = -Ae^{-\pi}$ on I_3 and similarly $C_2 = A$ on I_1 and $C_2 = -Ae^{\pi}$ I_3. Hence the answer to the problem is

$$f(x) = \begin{cases} Ae^{-x} & \text{for } x \in (-\infty, 0], \\ A(\sin x + \cos x) & \text{for } x \in (0, \pi], \\ -Ae^{-x+\pi} & \text{for } x \in (\pi, \infty), \end{cases}$$

$$g(x) = \begin{cases} Ae^{x} & \text{for } x \in (-\infty, 0], \\ A(\sin x - \cos x) & \text{for } x \in (0, \pi], \\ -Ae^{x-\pi} & \text{for } x \in (\pi, \infty), \end{cases}$$

where A is some nonzero constant.

(Romanian Mathematical Olympiad, 1976, proposed by V. Matrosenco)

555. The idea is to integrate the equation using an integrating factor. If instead we ha the first-order differential equation $(x^2 + y^2)dx + xydy = 0$, then the standard metho finds x as an integrating factor. So if we multiply our equation by f to transform it int

$$(f^3 + fg^2)f' + f^2gg' = 0,$$

then the new equation is equivalent to

$$\left(\frac{1}{4}f^4 + \frac{1}{2}f^2g^2\right)' = 0.$$

Therefore, f and g satisfy

$$f^4 + 2f^2g^2 = C,$$

for some real constant C. In particular, f is bounded.

(R. Gelca)

556. The idea is to write the equation as

$$Bydx + Axdy + x^m y^n (Dydx + Cxdy) = 0,$$

then find an integrating factor that integrates simultaneously $Bydx + Axdy$ and $x^m y^n (Dydx + Cxdy)$. An integrating factor of $Bydx + Axdy$ will be of the form $x^{-1}y^{-1}\phi_1(x^B y^A)$, while an integrating factor of $x^m y^n (Dydx + Cxdy) = Dx^m y^{n+1}dx + Cx^{m+1}y^n dy$ will be of the form $x^{-m-1}y^{-n-1}\phi_2(x^D y^C)$, where ϕ_1 and ϕ_2 are one-variable functions. To have the same integrating factor for both expressions, we should have

$$x^m y^n \phi_1(x^B y^A) = \phi_2(x^D y^C).$$

It is natural to try power functions, say $\phi_1(t) = t^p$ and $\phi_2(t) = t^q$. The equality condition gives rise to the system

$$Ap - Cq = -n,$$
$$Bp - Dq = -m,$$

which according to the hypothesis can be solved for p and q. We find that

$$p = \frac{Bn - Am}{AD - BC}, \quad q = \frac{Dn - Cm}{AD - BC}.$$

Multiplying the equation by $x^{-1}y^{-1}(x^B y^A)^p = x^{-1-m}y^{-1-n}(x^D y^C)^q$ and integrating, we obtain

$$\frac{1}{p+1}(x^B y^A)^{p+1} + \frac{1}{q+1}(x^D y^C)^{q+1} = \text{constant},$$

which gives the solution in implicit form.

(M. Ghermănescu, *Ecuaţii Diferenţiale* (*Differential Equations*), Editura Didactică şi Pedagogică, Bucharest, 1963)

557. The differential equation can be rewritten as

$$e^{y' \ln y} = e^{\ln x}.$$

Because the exponential function is injective, this is equivalent to $y' \ln y = \ln x$. Integrating, we obtain the algebraic equation $y \ln y - y = x \ln x - x + C$, for some constant C. The initial condition yields $C = 0$. We are left with finding all differentiable functions y such that

$$y \ln y - y = x \ln x - x.$$