Analysis vs. Algebra; the brute-force attack

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1 The problem

Here is a "classical" problem:

Claim 1.1 *The function* $f(x) = x^\alpha$, $x \in [0, \infty)$ *is* α*-Hölder.*

Definition 1.2 *Let* $\Omega \subset \mathbb{R}^n$ *be a (nonempty) set and* $0 < \alpha \leq 1$.

A function $f : \Omega \to \mathbb{R}$ *is* **Hölder with exponent** α *if there is a constant* $M > 0$ *such that* $|f(x) - f(y)| \leq M\|x - y\|^\alpha$ *for any* $x, y \in \Omega$.

[For $\alpha = 1$ *the function is usually called Lipschitz.]*

Here $\|x\|$ is the usual distance in $\mathbb{R}^n$, e.g. in $\mathbb{R}^2$ it is $\|(x, y)\| = \sqrt{x^2 + y^2}$.

Remark 1.3 *If* $\alpha > 1$ *then the only such functions on, e.g.,* $\Omega = [0, 1] \subset \mathbb{R}$*, are the constant functions. This is also solvable with Calculus I.*

So, to prove the Claim, we have to find an $M > 0$ such

$$|x^\alpha - y^\alpha| \leq M|x - y|^\alpha \text{ for any } x, y \in [0, \infty). \tag{1.1}$$

Equivalently, that

$$|x^\alpha - y^\alpha| - M|x - y|^\alpha \leq 0 \text{ for any } x, y \in [0, \infty) \tag{1.2}$$

or

$$\frac{|x^\alpha - y^\alpha|}{|x - y|^\alpha} \leq M \text{ for any } x, y \in [0, \infty), x \neq y \tag{1.3}$$

(for $x = y$ relation (1.3) does not make sense but clearly (1.1) is true).

2 What algebra can do

For $\alpha = 1/2$ one can find a bound M in (1.2) or (1.3) with algebraic manipulations (in (1.1) take powers, in (1.3) multiply both numerator and denominator the "conjugate", $\sqrt{x} + \sqrt{y}$).

Similarly for $\alpha = p/q$, rational.

One could try to get an explicit bound M_α for α rational, then take a limit.

Calculus is a more powerful tool!

3 Finding extreme values

Remember from Calculus I (idea is similar for functions of more variables, Calculus III):

Theorem 3.1 (Extreme value theorem, EVT) *Let $\Omega \in \mathbb{R}^n$ be a* **closed and bounded** *set*[1] *and $f : \Omega \to \mathbb{R}$ a* **continuous** *function.*

Then

- *(a) f is bounded on Ω (just what we need in (1.3));*
- *(b) there are points in Ω where f reaches its maximum, respectively its minimum.*

Theorem 3.2 (Fermat's Theorem) *Let $\Omega \subset \mathbb{R}^n$ be an* **open** *set, and $f : \Omega \to R$ a function. If $x \in \Omega$ is a (local) extremum*[2] *of f, then*

- *(a) either f is not differentiable at x*
- *(b) or $f'(x) = 0$ [respectively, for functions of more variables, all partial derivatives are zero at x].*

This gives a way to find extrema of functions.

WARNING: Let $f(x) := e^{-x}$ for $x \in [0, \infty)$. Then $0 \leq f(x) \leq 1$, but cannot just apply the above theorems. We have to take into account the "boundary" of the domain, here 0 and ∞.

In higher dimensions this could be more complicated: e.g., find the extrema of $f(x, y) = x^2 - 2y^2$ on the closed unit disk, $\Omega = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$.

4 Back to our claim

As (1.2) is stated, we have to find an unknown M. It is more convenient to look at (1.3): we only have to show that

$$F(x, y) := \frac{|x^\alpha - y^\alpha|}{|x - y|^\alpha}$$

is bounded from above for $x, y \geq 0$, $x \neq y$.

The EVT cannot be used. Why not?

4.1 Simplifying the problem

- Note first that it is enough to solve the case $x < y$. This is still a problem in two variables.
- **MAIN SIMPLIFICATION:** Notice that $F(tx, ty) = F(x, y)$ for $t > 0$.

Therefore, if we denote $t = x/y$, so $x = ty$, we get

$$F(x, y) = \frac{|t^\alpha - 1|}{|t - 1|^\alpha} = h(t)$$

and we now only have to consider a function $h(t)$ of a single variable, $t \in (0, 1)$. This domain at least is bounded!

Note that one can write $h(t)$ without the absolute values if $0 < t < 1$.

Still cannot apply EVT! Why not?

4.2 ASSIGNMENT

Fill in the steps so that the EVT can be used to claim that there is an upper bound for $h(t)$ on $(0, 1)$.

[1] Such sets are called ***compact***, as is clarified in more advances courses.

[2] That is, minimum or maximum.