William Lowell Putnam Mathematical Competition
Problems B1, 1985-2023 (newest first from 2010)

B1 ('23) Consider an m-by-n grid of unit squares, indexed by (i,j) with $1 \leq i \leq m$ and $1 \leq j \leq n$. There are $(m-1)(n-1)$ coins, which are initially placed in the squares (i,j) with $1 \leq i \leq m-1$ and $1 \leq j \leq n-1$. If a coin occupies the square (i,j) with $i \leq m-1$ and $j \leq n-1$ and the squares $(i+1,j)$, $(i,j+1)$, and $(i+1,j+1)$ are unoccupied, then a legal move is to slide the coin from (i,j) to $(i+1,j+1)$. How many distinct configurations of coins can be reached starting from the initial configuration by a (possibly empty) sequence of legal moves?

B1 ('22) Suppose that $P(x) = a_1x + a_2x^2 + \cdots + a_nx^n$ is a polynomial with integer coefficients, with a_1 odd. Suppose that $e^{P(x)} = b_0 + b_1x + b_2x^2 + \cdots$ for all x. Prove that b_k is nonzero for all $k \geq 0$.

B1 ('21) Suppose that the plane is tiled with an infinite checkerboard of unit squares. If another unit square is dropped on the plane at random with position and orientation independent of the checkerboard tiling, what is the probability that it does not cover any of the corners of the squares of the checkerboard?

B1 ('20) For a positive integer n, define $d(n)$ to be the sum of the digits of n when written in binary (for example, $d(13) = 1+1+0+1 = 3$). Let

$$S = \sum_{k=1}^{2020} (-1)^{d(k)} k^3.$$

Determine S modulo 2020.

B1 ('19) Denote by $\mathbb{Z}^2$ the set of all points (x,y) in the plane with integer coordinates. For each integer $n \geq 0$, let P_n be the subset of $\mathbb{Z}^2$ consisting of the point $(0,0)$ together with all points (x,y) such that $x^2 + y^2 = 2^k$ for some integer $k \leq n$. Determine, as a function of n, the number of four-point subsets of P_n whose elements are the vertices of a square.

B1 ('18) Let $\mathcal{P}$ be the set of vectors defined by

$$\mathcal{P} = \left\{ \begin{pmatrix} a \\ b \end{pmatrix} \middle| \, 0 \leq a \leq 2, 0 \leq b \leq 100, \text{ and } a, b \in \mathbb{Z} \right\}.$$

Find all $\mathbf{v} \in \mathcal{P}$ such that the set $\mathcal{P} \setminus \{\mathbf{v}\}$ obtained by omitting vector $\mathbf{v}$ from $\mathcal{P}$ can be partitioned into two sets of equal size and equal sum.

B1 ('17) Let L_1 and L_2 be distinct lines in the plane. Prove that L_1 and L_2 intersect if and only if, for every real number $\lambda \neq 0$ and every point P not on L_1 or L_2, there exist points A_1 on L_1 and A_2 on L_2 such that $\overrightarrow{PA_2} = \lambda \overrightarrow{PA_1}$.

B1 ('16) Let $x_0, x_1, x_2, \ldots$ be the sequence such that $x_0 = 1$ and for $n \geq 0$,

$$x_{n+1} = \ln(e^{x_n} - x_n)$$

(as usual, the function $\ln$ is the natural logarithm). Show that the infinite series

$$x_0 + x_1 + x_2 + \cdots$$

converges and find its sum.

B1 ('15) Let f be a three times differentiable function (defined on $\mathbb{R}$ and real-valued) such that f has at least five distinct real zeros. Prove that $f + 6f' + 12f'' + 8f'''$ has at least two distinct real zeros.

B1 ('14) A *base* 10 *over-expansion* of a positive integer N is an expression of the form

$$N = d_k10^k + d_{k-1}10^{k-1} + \cdots + d_010^0$$

with $d_k \neq 0$ and $d_i \in \{0,1,2,\ldots,10\}$ for all i. For instance, the integer $N = 10$ has two base 10 over-expansions: $10 = 10 \cdot 10^0$ and the usual base 10 expansion $10 = 1 \cdot 10^1 + 0 \cdot 10^0$. Which positive integers have a unique base 10 over-expansion?

B1 ('13) For positive integers n, let the numbers $c(n)$ be determined by the rules $c(1) = 1$, $c(2n) = c(n)$, and $c(2n+1) = (-1)^n c(n)$. Find the value of

$$\sum_{n=1}^{2013} c(n)c(n+2).$$

B1 ('12) Let S be a class of functions from $[0,\infty)$ to $[0,\infty)$ that satisfies:

(i) The functions $f_1(x) = e^x - 1$ and $f_2(x) = \ln(x+1)$ are in S;

(ii) If $f(x)$ and $g(x)$ are in S, the functions $f(x) + g(x)$ and $f(g(x))$ are in S;

(iii) If $f(x)$ and $g(x)$ are in S and $f(x) \geq g(x)$ for all $x \geq 0$, then the function $f(x) - g(x)$ is in S.

Prove that if $f(x)$ and $g(x)$ are in S, then the function $f(x)g(x)$ is also in S.

B1 ('11) Let h and k be positive integers. Prove that for every $\varepsilon > 0$, there are positive integers m and n such that

$$\varepsilon < \left| h\sqrt{m} - k\sqrt{n} \right| < 2\varepsilon.$$

B1 ('10) Is there an infinite sequence of real numbers $a_1, a_2, a_3, \ldots$ such that

$$a_1^m + a_2^m + a_3^m + \cdots = m$$

for every positive integer m?

B1 ('85) Let k be the smallest positive integer for which there exist distinct integers m_1, m_2, m_3, m_4, m_5 such that the polynomial

$$p(x) = (x - m_1)(x - m_2)(x - m_3)(x - m_4)(x - m_5)$$

has exactly k nonzero coefficients. Find, with proof, a set of integers m_1, m_2, m_3, m_4, m_5 for which this minimum k is achieved.

B1 ('86) Inscribe a rectangle of base b and height h in a circle of radius one, and inscribe an isosceles triangle in the region of the circle cut off by one base of the rectangle (with that side as the base of the triangle). For what value of h do the rectangle and triangle have the same area?

B1 ('87) Evaluate

$$\int_2^4 \frac{\sqrt{\ln(9-x)}\,dx}{\sqrt{\ln(9-x)} + \sqrt{\ln(x+3)}}.$$

B1 ('88) A *composite* (positive integer) is a product ab with a and b not necessarily distinct integers in $\{2, 3, 4, \dots\}$. Show that every composite is expressible as $xy + xz + yz + 1$, with x, y, z positive integers.

B1 ('89) A dart, thrown at random, hits a square target. Assuming that any two parts of the target of equal area are equally likely to be hit, find the probability that the point hit is nearer to the center than to any edge. Express your answer in the form $\dfrac{a\sqrt{b}+c}{d}$, where a, b, c, d are integers.

B1 ('90) Find all real-valued continuously differentiable functions f on the real line such that for all x,

$$(f(x))^2 = \int_0^x [(f(t))^2 + (f'(t))^2]\,dt + 1990.$$

B1 ('91) For each integer $n \geq 0$, let $S(n) = n - m^2$, where m is the greatest integer with $m^2 \leq n$. Define a sequence $(a_k)_{k=0}^{\infty}$ by $a_0 = A$ and $a_{k+1} = a_k + S(a_k)$ for $k \geq 0$. For what positive integers A is this sequence eventually constant?

B1 ('92) Let S be a set of n distinct real numbers. Let A_S be the set of numbers that occur as averages of two distinct elements of S. For a given $n \geq 2$, what is the smallest possible number of elements in A_S?

B1 ('93) Find the smallest positive integer n such that for every integer m with $0 < m < 1993$, there exists an integer k for which

$$\frac{m}{1993} < \frac{k}{n} < \frac{m+1}{1994}.$$

B1 ('94) Find all positive integers n that are within 250 of exactly 15 perfect squares.

B1 ('95) For a partition π of $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$, let $\pi(x)$ be the number of elements in the part containing x. Prove that for any two partitions π and π', there are two distinct numbers x and y in $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ such that $\pi(x) = \pi(y)$ and $\pi'(x) = \pi'(y)$. [A *partition* of a set S is a collection of disjoint subsets (parts) whose union is S.]

B1 ('96) Define a **selfish** set to be a set which has its own cardinality (number of elements) as an element. Find, with proof, the number of subsets of $\{1, 2, \dots, n\}$ which are *minimal* selfish sets, that is, selfish sets none of whose proper subsets is selfish.

B1 ('97) Let $\{x\}$ denote the distance between the real number x and the nearest integer. For each positive integer n, evaluate

$$F_n = \sum_{m=1}^{6n-1} \min(\{\frac{m}{6n}\}, \{\frac{m}{3n}\}).$$

(Here $\min(a, b)$ denotes the minimum of a and b.)

B1 ('98) Find the minimum value of

$$\frac{(x+1/x)^6 - (x^6 + 1/x^6) - 2}{(x+1/x)^3 + (x^3 + 1/x^3)}$$

for $x > 0$.

B1 ('99) Right triangle ABC has right angle at C and $\angle BAC = \theta$; the point D is chosen on AB so that $|AC| = |AD| = 1$; the point E is chosen on BC so that $\angle CDE = \theta$. The perpendicular to BC at E meets AB at F. Evaluate $\lim_{\theta \to 0} |EF|$.

B1 ('00) Let a_j, b_j, c_j be integers for $1 \leq j \leq N$. Assume for each j, at least one of a_j, b_j, c_j is odd. Show that there exist integers r, s, t such that $ra_j + sb_j + tc_j$ is odd for at least $4N/7$ values of j, $1 \leq j \leq N$.

B1 ('01) Let n be an even positive integer. Write the numbers $1, 2, \dots, n^2$ in the squares of an $n \times n$ grid so that the k-th row, from left to right, is

$$(k-1)n + 1, (k-1)n + 2, \dots, (k-1)n + n.$$

Color the squares of the grid so that half of the squares in each row and in each column are red and the other half are black (a checkerboard coloring is one possibility). Prove that for each coloring, the sum of the numbers on the red squares is equal to the sum of the numbers on the black squares.

B1 ('02) Shanille O'Keal shoots free throws on a basketball court. She hits the first and misses the second, and thereafter the probability that she hits the next shot is equal to the proportion of shots she has hit so far. What is the probability she hits exactly 50 of her first 100 shots?

B1 ('03) Do there exist polynomials $a(x), b(x), c(y), d(y)$ such that

$$1 + xy + x^2y^2 = a(x)c(y) + b(x)d(y)$$

holds identically?

B1 ('04) Let $P(x) = c_nx^n + c_{n-1}x^{n-1} + \cdots + c_0$ be a polynomial with integer coefficients. Suppose that r is a rational number such that $P(r) = 0$. Show that the n numbers

$$c_nr,\ c_nr^2 + c_{n-1}r,\ c_nr^3 + c_{n-1}r^2 + c_{n-2}r,\\ \ldots,\ c_nr^n + c_{n-1}r^{n-1} + \cdots + c_1r$$

are integers.

B1 ('05) Find a nonzero polynomial $P(x, y)$ such that $P(\lfloor a\rfloor, \lfloor 2a\rfloor) = 0$ for all real numbers a. (Note: $\lfloor \nu\rfloor$ is the greatest integer less than or equal to ν.)

B1 ('06) Show that the curve $x^3 + 3xy + y^3 = 1$ contains only one set of three distinct points, A, B, and C, which are vertices of an equilateral triangle, and find its area.

B1 ('07) Let f be a polynomial with positive integer coefficients. Prove that if n is a positive integer, then $f(n)$ divides $f(f(n) + 1)$ if and only if $n = 1$. [Editor's note: one must assume f is nonconstant.]

B1 ('08) What is the maximum number of rational points that can lie on a circle in $\mathbb{R}^2$ whose center is not a rational point? (A *rational point* is a point both of whose coordinates are rational numbers.)

B1 ('09) Show that every positive rational number can be written as a quotient of products of factorials of (not necessarily distinct) primes. For example,

$$\frac{10}{9} = \frac{2! \cdot 5!}{3! \cdot 3! \cdot 3!}.$$