Here is a selection of problems from the sections covered by the third exam. Some of the problems were already assigned as homework.

Note that some of the problems can be solved with more than one method (as we progressed in the book, we learned more general ways to deal with various topics). Thus, even if the problem is listed in a particular section, you might use a more advanced method to solve it.

§5.6: 5, 9 (do by hand)

§6.1: 17

§6.2, §6.5: Solve the system $X' = CX, X(0) = (1,2)^t$ for the following cases. Try both the direct and matrix exponential method.

(a) $C = \begin{pmatrix} -7 & 9 \\ -6 & 8 \end{pmatrix}$

(b) $C = \begin{pmatrix} 4 & -1 \\ 1 & 2 \end{pmatrix}$

(c) $C = \begin{pmatrix} -5 & 10 \\ -5 & 9 \end{pmatrix}$

§6.3: 4, 5, 6, 9

§6.4: 1, 2 (try both the direct and matrix exponential method; the latter is simpler for these problems)

§6.5: 1, 3, 4, 7

- Are the matrices $C = \begin{pmatrix} 8 & -3 \\ 10 & -3 \end{pmatrix}$ and $D = \begin{pmatrix} 11 & -6 \\ 12 & -6 \end{pmatrix}$ similar? If yes, find a matrix P such that $PCP^{-1} = D$.

§8.1: 7

§8.2: 7, 8, 9, 10

Some Answers

§6.2-§6.5: Solve the system $\mathbf{X}' = C\mathbf{X}$, $\mathbf{X}(0) = (1,2)^t$, and compute e^{tC}, for:

(a) $C = \begin{pmatrix} -7 & 9 \\ -6 & 8 \end{pmatrix}$

Answer:

- eigenvalues: $\lambda_1 = 2$, $\lambda_2 = -1$
- eigenvectors: $\mathbf{v}_1 = (1,1)^t$, $\mathbf{v}_2 = (3,2)^t$
- general solution: $\mathbf{X}(t) = \alpha e^{\lambda_1 t}\mathbf{v}_1 + \beta e^{\lambda_2 t}\mathbf{v}_2$
- IVP: $\alpha = 4$, $\beta = -1$ hence

$$\mathbf{X}(t) = \begin{pmatrix} 4e^{2t} - 3e^{-t} \\ 4e^{2t} - 2e^{-t} \end{pmatrix}$$

- e^{tC}:

$$P = \begin{pmatrix} 1 & 3 \\ 1 & 2 \end{pmatrix}, \quad C = P \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix} P^{-1},$$

$$\implies e^{tC} = P \begin{pmatrix} e^{2t} & 0 \\ 0 & e^{-t} \end{pmatrix} P^{-1} = \begin{pmatrix} -2e^{2t} + 3e^{-t} & 3e^{2t} - 3e^{-t} \\ -2e^{2t} + 2e^{-t} & 3e^{2t} - 2e^{-t} \end{pmatrix}$$

(b) $C = \begin{pmatrix} 4 & -1 \\ 1 & 2 \end{pmatrix}$

Answer:

- eigenvalues: $\lambda_1 = \lambda_2 = 3$
- eigenvectors: $\mathbf{v}_1 = (1,1)^t$, generalized eigenvector $\mathbf{w}_1 = (1,0)^t$ (this is not unique)
- general solution: $\mathbf{X}(t) = \alpha e^{3t}\mathbf{v}_1 + \beta e^{3t}(\mathbf{w}_1 + t\mathbf{v}_1)$
- IVP: $\alpha = 2$, $\beta = -1$ hence

$$\mathbf{X}(t) = e^{3t} \begin{pmatrix} 1 - t \\ 2 - t \end{pmatrix}$$

- e^{tC}:

$$P = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \quad C = P \begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix} P^{-1},$$
$$\Longrightarrow e^{tC} = P \left[e^{3t} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \right] P^{-1} = e^{3t} \begin{pmatrix} 1+t & -t \\ t & 1-t \end{pmatrix}$$

(c) $C = \begin{pmatrix} -5 & 10 \\ -5 & 9 \end{pmatrix}$

Answer:

- eigenvalues: $\lambda_1 = 2 + i$, $\lambda_2 = 2 - i$
- eigenvectors: $\mathbf{v}_1 = (7 - i, 5)^t \quad \Longrightarrow \quad \mathbf{v}_2 = (7 + i, 5)^t = (7, 5)^t + i(1, 0)^t$
  [Use $\sigma = 2, \tau = 1$. Note that P is constructed using $\mathbf{v}_2$!]
- general solution:

$$\mathbf{X}(t) = \alpha \operatorname{Re}(e^{\lambda_1 t}\mathbf{v}_1) + \beta \operatorname{Im}(e^{\lambda_1 t}\mathbf{v}_1) = e^{2t} \left(\alpha \begin{pmatrix} 7\cos t + \sin t \\ 5\cos t \end{pmatrix} + \beta \begin{pmatrix} 7\sin t - \cos t \\ 5\sin t \end{pmatrix} \right)$$

- IVP: $\alpha = 2/5$, $\beta = 9/5$, hence

$$\mathbf{X}(t) = e^{2t} \begin{pmatrix} \cos t + 13\sin t \\ 2\cos + 9\sin t \end{pmatrix}$$

- e^{tC}:

$$P = \begin{pmatrix} 7 & 1 \\ 5 & 0 \end{pmatrix}, \quad C = P \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix} P^{-1},$$
$$\Longrightarrow e^{tC} = P \left[e^{2t} \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \right] P^{-1} = e^{2t} \begin{pmatrix} \cos t - 7\sin t & 10\sin t \\ -5\sin t & \cos t + 7\sin t \end{pmatrix}$$