

Math 2431

Here is a list of some topics.

- vectors in \mathbb{R}^n , matrices
 - addition, multiplication by a scalar
 - product of matrices, and matrix \times vector; properties (linear maps = matrix maps)
 - norm and dot product of vectors, angles and areas
- systems of linear equations: $Ax = b$
 - row echelon form via Gauss elimination
 - reduced row echelon form is unique
 - finding solutions (a basis of solutions if $b = 0$)
 - rank
 - consistent and inconsistent systems
 - if consistent: # of param's = # unknowns – rank A
 - if $b = 0$, dimension of solution space = # unknowns – rank A
 - superposition (§3.4)
 - solutions for low-dimensional systems
- matrices, linearity, inverses
 - linear maps (= matrix maps)
 - linear maps in \mathbb{R}^2
 - superposition (§3.4, §4.7)
 - computing inverses (via row reduction)
 - solving $Ax = b$ with A^{-1}
 - 2 x 2 determinants and inverses; $\text{area}(A(P)) = |\det(A)| \text{area}(P)$
- systems of ODE's
 - one ODE, initial value problems ($x' = f(x, t)$, $x(t_0) = x_0$)
 - autonomous ODE's ($x' = g(x)$); equilibria ($g(x) = 0$) and stability of hyperbolic equilibria ($g'(x_0) \neq 0$)
 - $x' = \lambda x \implies x(t) = x_0 e^{\lambda t}$
 - 2 x 2 systems (see much more in Chapter 6); sinks, sources, saddles
 - graphic representations: time-series and phase-space portraits
- vector spaces
 - definition
 - subspaces (V vector space, $W \subset V$; W subspace $\iff W$ closed under addition and multiplication by scalars)
e.g.: $\text{null}(A) = \{x \in \mathbb{R}^n \mid Ax = 0\}$, $\{X \in (\mathcal{C}^1)^n \mid X' = CX\}$
 - span of a family of vectors
 - spanning sets
 - linear independence
 - dimension, bases
Theorem 5.5.3, Corollary 5.6.7

- dimension of a span and of a null-space:
 $\dim(\text{span}\{w_1, \dots, w_k\}) = \text{rank}(M^t) = \text{rank}(M)$, $M = (w_1 | \dots | w_k)$
 $\dim(\text{null}(A)) + \text{rank}(A) = \# \text{ of columns (i.e., } \# \text{ of variables)}$
- planar ODE's: $X' = CX$, C an $n \times n$ matrix
 - ASIDE. general ODE's: $X' = F(X)$; F has continuous derivatives
 \implies uniqueness of solutions to IVP
 - dimension of space of solutions = n (size of C), hence need n linearly independent solutions and use superposition
 - solutions $\{X_1(t), \dots, X_n(t)\}$ are linearly independent if so is the family of vectors $\{X_1(0), \dots, X_n(0)\}$
 - direct method and matrix exponentiation
 - direct method (§6.2): three cases
 - * λ_1, λ_2 real, two linearly independent eigenvectors
 - * λ_1, λ_2 complex conjugate
 - * $\lambda_1 = \lambda_2$ real, one linearly independent eigenvector (use generalized eigenvector)
 - matrix exponentiation: find similarity to normal form (Thm. 6.5.5)
- determinants and eigenvalues (for $n \times n$ matrices)
 - properties: $\det(A)$ for A triangular, $\det(A) = \det(A^t)$, $\det(AB) = \det(A)\det(B)$
 - computing determinants: reduction to row echelon form or cofactor expansion
 - A invertible $\iff \det(A) \neq 0$
 - characteristic polynomial $p_A(\lambda) = \det(A - \lambda I)$
 - eigenvalues: roots of $p_A(\lambda) = 0$
 - eigenvectors: nonzero vectors with $Av = \lambda v$ (hence λ is an eigenvalue)
 - $\det(A) = \lambda_1 \cdot \lambda_2 \cdot \dots \cdot \lambda_n$, $\text{trace}(A) = \lambda_1 + \dots + \lambda_n$
 - $\det(A) \neq 0 \iff$ no eigenvalue equal to zero
 - A invertible \implies eigenvalues of A^{-1} are the inverses of the eigenvalues of A
 - A and B similar \implies they have the same char. polynomial, hence the same eigenvalues

What we should have also covered:

- * second order ODE's (§6.7)
- * qualitative theory of planar ODE's (Ch. 7): $X' = CX$
 - stability of equilibria
 - all $\text{Re}(\lambda_k) < 0 \implies$ origin asymptotically stable; one $\text{Re}(\lambda_k) > 0 \implies$ origin unstable
 - hyperbolic systems: all $\text{Re}(\lambda_k) \neq 0$
 - classification in two dimensions:
 - * hyperbolic systems:
 - saddle: $\lambda_1 \lambda_2 = \det(C) < 0$
 - sinks (and sources): spiral (λ_k complex), nodal ($\lambda_1 \neq \lambda_2$ real), improper nodal ($\lambda_1 = \lambda_2$ real, 1 eigenvector), focus ($\lambda_1 = \lambda_2$ real, 2 eigenvectors, i.e. $C = \lambda I_2$)
 - * nonhyperbolic systems: center ($\lambda_{1,2} = \pm i\tau$), saddle node ($\lambda_1 = 0, \lambda_2 \neq 0$), shear ($\lambda_1 = \lambda_2 = 0, 1$ eigenvector), $C = \text{zero matrix}$