Here is a selection of problems from the sections covered by the third exam. Some of the problems were already assigned as homework.

§5.4: 3, 5

§5.5: 1, 2, 3

§5.6: 5, 9 (do by hand)

§6.1: 17

§6.2, §6.5: Solve the system $X' = CX, X(0) = (1, 2)^t$ for the following cases. Try both the direct method, and computing the exponential e^{tC}.

(a) $C = \begin{pmatrix} -7 & 9 \\ -6 & 8 \end{pmatrix}$

(b) $C = \begin{pmatrix} 4 & -1 \\ 1 & 2 \end{pmatrix}$

(c) $C = \begin{pmatrix} -5 & 10 \\ -5 & 9 \end{pmatrix}$

§6.3: 4, 5, 6, 9

§6.4: 1, 2 (try both the direct and matrix exponential method; the latter is simpler for these problems)

§6.5: 1, 3, 4, 7

- Are the matrices $C = \begin{pmatrix} 8 & -3 \\ 10 & -3 \end{pmatrix}$ and $D = \begin{pmatrix} 11 & -6 \\ 12 & -6 \end{pmatrix}$ similar? If yes, find a matrix P such that $PCP^{-1} = D$.

§8.1: 7

§8.2: 7, 8, 9, 10

Some Answers

§6.2-§6.5: Solve the system $\mathbf{X}' = C\mathbf{X}$, $\mathbf{X}(0) = (1,2)^t$, and compute e^{tC}, for:

(a) $C = \begin{pmatrix} -7 & 9 \\ -6 & 8 \end{pmatrix}$

Answer:

- eigenvalues: $\lambda_1 = 2$, $\lambda_2 = -1$
- eigenvectors: $\mathbf{v}_1 = (1,1)^t$, $\mathbf{v}_2 = (3,2)^t$
- general solution: $\mathbf{X}(t) = \alpha e^{\lambda_1 t}\mathbf{v}_1 + \beta e^{\lambda_2 t}\mathbf{v}_2$
- IVP: $\alpha = 4$, $\beta = -1$ hence

$$\mathbf{X}(t) = \begin{pmatrix} 4e^{2t} - 3e^{-t} \\ 4e^{2t} - 2e^{-t} \end{pmatrix}$$

- e^{tC}:

$$P = \begin{pmatrix} 1 & 3 \\ 1 & 2 \end{pmatrix}, \quad C = P\begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}P^{-1},$$

$$\implies e^{tC} = P\begin{pmatrix} e^{2t} & 0 \\ 0 & e^{-t} \end{pmatrix}P^{-1} = \begin{pmatrix} -2e^{2t} + 3e^{-t} & 3e^{2t} - 3e^{-t} \\ -2e^{2t} + 2e^{-t} & 3e^{2t} - 2e^{-t} \end{pmatrix}$$

(b) $C = \begin{pmatrix} 4 & -1 \\ 1 & 2 \end{pmatrix}$

Answer:

- eigenvalues: $\lambda_1 = \lambda_2 = 3$
- eigenvectors: $\mathbf{v}_1 = (1,1)^t$, generalized eigenvector $\mathbf{w}_1 = (1,0)^t$ (this is not unique)
- general solution: $\mathbf{X}(t) = \alpha e^{3t}\mathbf{v}_1 + \beta e^{3t}(\mathbf{w}_1 + t\mathbf{v}_1)$
- IVP: $\alpha = 2$, $\beta = -1$ hence

$$\mathbf{X}(t) = e^{3t}\begin{pmatrix} 1 - t \\ 2 - t \end{pmatrix}$$

- e^{tC}:

$$P = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \quad C = P \begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix} P^{-1},$$
$$\implies e^{tC} = P\left[e^{3t} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \right] P^{-1} = e^{3t} \begin{pmatrix} 1+t & -t \\ t & 1-t \end{pmatrix}$$

(c) $C = \begin{pmatrix} -5 & 10 \\ -5 & 9 \end{pmatrix}$

Answer:

- eigenvalues: $\lambda_1 = 2+i$, $\lambda_2 = 2-i$
- eigenvectors: $\mathbf{v}_1 = (7-i, 5)^t \quad \implies \quad \mathbf{v}_2 = (7+i, 5)^t = (7,5)^t + i(1,0)^t$

  [Use $\sigma = 2, \tau = 1$. Note that P is constructed using $\mathbf{v}_2$!]
- general solution:

$$\mathbf{X}(t) = \alpha \operatorname{Re}(e^{\lambda_1 t}\mathbf{v}_1) + \beta \operatorname{Im}(e^{\lambda_1 t}\mathbf{v}_1) = e^{2t}\left(\alpha \begin{pmatrix} 7\cos t + \sin t \\ 5\cos t \end{pmatrix} + \beta \begin{pmatrix} 7\sin t - \cos t \\ 5\sin t \end{pmatrix} \right)$$

- IVP: $\alpha = 2/5$, $\beta = 9/5$, hence

$$\mathbf{X}(t) = e^{2t} \begin{pmatrix} \cos t + 13\sin t \\ 2\cos + 9\sin t \end{pmatrix}$$

- e^{tC}:

$$P = \begin{pmatrix} 7 & 1 \\ 5 & 0 \end{pmatrix}, \quad C = P \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix} P^{-1},$$
$$\implies e^{tC} = P\left[e^{2t} \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \right] P^{-1} = e^{2t} \begin{pmatrix} \cos t - 7\sin t & 10\sin t \\ -5\sin t & \cos t + 7\sin t \end{pmatrix}$$