Math 2431

Here are some of the topics discussed in this course:

- vectors in $\mathbb{R}^n$, matrices
  - addition, multiplication by a scalar
  - product of matrices, and matrix $\times$ vector; properties (linear maps = matrix maps)
  - norm and dot product of vectors, angles and areas
- systems of linear equations: $Ax = b$
  - row echelon form via Gauss elimination
  - reduced row echelon form is unique
  - finding solutions (a basis of solutions if $b = 0$)
  - rank
  - consistent and inconsistent systems
  - if consistent: # of param's = # unknowns $-\operatorname{rank} A$
  - if $b = 0$, dimension of solution space = # unknowns $-\operatorname{rank} A$
  - superposition (§3.4)
  - solutions for low-dimensional systems
- matrices, linearity, inverses
  - linear maps (= matrix maps)
  - linear maps in $\mathbb{R}^2$
  - superposition (§3.4, §4.7)
  - computing inverses (via row reduction)
  - solving $Ax = b$ with A^{-1}
  - 2 x 2 determinants and inverses; $\operatorname{area}(A(P)) = |\det(A)| \operatorname{area}(P)$
- systems of ODE's
  - one ODE, initial value problems ($x' = f(x, t),\ x(t_0) = x_0$)
  - autonomous ODE's ($x' = g(x)$); equilibria ($g(x) = 0$) and stability of hyperbolic equilibria ($g'(x_0) \neq 0$)
  - $x' = \lambda x \implies x(t) = x_0 e^{\lambda t}$
  - 2 x 2 systems (see much more in Chapter 6); sinks, sources, saddles
  - graphic representations: time-series and phase-space portraits
- vector spaces
  - definition
  - subspaces (V vector space, $W \subset V$; W subspace $\iff$ W closed under addition and multiplication by scalars)
    e.g.: $\operatorname{null}(A) = \{x \in \mathbb{R}^n \mid Ax = 0\}$, $\{X \in (\mathcal{C}^1)^n \mid X' = CX\}$
  - span of a family of vectors
  - spanning sets
  - linear independence
  - dimension, bases
    Theorem 5.5.3, Corollary 5.6.7

- dimension of a span and of a null-space:
  $\dim(\text{span}\{w_1, \ldots, w_k\}) = \text{rank}(M^t) = \text{rank}(M)$, $M = (w_1|\ldots|w_k)$
  $\dim(\text{null}(A)) + \text{rank}(A) = \#$ of columns (i.e., # of variables)

- planar ODE's: $X' = CX$, C an $n \times n$ matrix
  - ASIDE. general ODE's: $X' = F(X)$; F has continuous derivatives $\Longrightarrow$ uniqueness of solutions to IVP
  - dimension of space of solutions = n (size of C), hence need n linearly independent solutions and use superposition
  - solutions $\{X_1(t), \ldots, X_n(t)\}$ are linearly independent if so is the family of vectors $\{X_1(0), \ldots, X_n(0)\}$
  - direct method and matrix exponentiation
  - direct method (§6.2): three cases
    * λ_1, λ_2 real, two linearly independent eigenvectors
    * λ_1, λ_2 complex conjugate
    * $\lambda_1 = \lambda_2$ real, one linearly independent eigenvector (use generalized eigenvector)
  - matrix exponentiation: find similarity to normal form (Thm. 6.5.5)

- determinants and eigenvalues (for $n \times n$ matrices)
  - properties: $\det(A)$ for A triangular, $\det(A) = \det(A^t)$, $\det(AB) = \det(A)\det(B)$
  - computing determinants: reduction to row echelon form or cofactor expansion
  - A invertible $\Longleftrightarrow$ $\det(A) \neq 0$
  - characteristic polynomial $p_A(\lambda) = \det(A - \lambda I)$
  - eigenvalues: roots of $p_A(\lambda) = 0$
  - eigenvectors: nonzero vectors with $Av = \lambda v$ (hence λ is an eigenvalue)
  - $\det(A) = \lambda_1 \cdot \lambda_2 \cdot \ldots \cdot \lambda_n$, $\text{trace}(A) = \lambda_1 + \cdots + \lambda_n$
  - $\det(A) \neq 0$ $\Longleftrightarrow$ no eigenvalue equal to zero
  - A invertible $\Longrightarrow$ eigenvalues of A^{-1} are the inverses of the eigenvalues of A
  - A and B similar $\Longrightarrow$ they have the same char. polynomial, hence the same eigenvalues

What we should have also covered:

* second order ODE's (§6.7)

* qualitative theory of planar ODE's (Ch. 7): $X' = CX$
  - stability of equilibria
  - all $Re(\lambda_k) < 0$ $\Longrightarrow$ origin asymptotically stable; one $Re(\lambda_k) > 0$ $\Longrightarrow$ origin unstable
  - hyperbolic systems: all $Re(\lambda_k) \neq 0$
  - classification in two dimensions:
    * hyperbolic systems:
      · saddle: $\lambda_1\lambda_2 = \det(C) < 0$
      · sinks (and sources): spiral (λ_k complex), nodal ($\lambda_1 \neq \lambda_2$ real), improper nodal ($\lambda_1 = \lambda_2$ real, 1 eigenvector), focus ($\lambda_1 = \lambda_2$ real, 2 eigenvectors, i.e. $C = \lambda I_2$)
    * nonhyperbolic systems: center ($\lambda_{1,2} = \pm i\tau$), saddle node ($\lambda_1 = 0, \lambda_2 \neq 0$), shear ($\lambda_1 = \lambda_2 = 0$, 1 eigenvector), C = zero matrix