

Chapter 12: VECTORS

1. **Geometry:** Let $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ be points in 3-space:

A. **Distance Formula:** $d(P_1, P_2) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$.

B. **Midpoint Formula:** The midpoint of the line segment joining P_1 and P_2 is the point

$$P \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2} \right)$$

C. **Equation for the sphere of radius r and center $P(a, b, c)$:**

$$(x - a)^2 + (y - b)^2 + (z - c)^2 = r^2$$

2. **Vectors:** A vector \mathbf{a} in n -dimensional space is an ordered n -tuple of real numbers. In particular, a vector \mathbf{a} in 3-space is an ordered triple of numbers: $\mathbf{a} = (a_1, a_2, a_3)$; a vector \mathbf{a} in the plane (2-space) is an ordered pair of numbers: $\mathbf{a} = (a_1, a_2)$. The vector $\mathbf{0} = (0, 0, 0)$ is the *zero vector*

Let $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{b} = (b_1, b_2, b_3)$ be vectors in 3-space and let α be a real number (scalar).

A. **Equality:** $\mathbf{a} = \mathbf{b}$ iff $a_1 = b_1, a_2 = b_2, a_3 = b_3$

B. **Vector Addition:** $\mathbf{a} + \mathbf{b} = (a_1 + b_1, a_2 + b_2, a_3 + b_3)$

C. **Multiplication by a Scalar:** $\alpha \mathbf{a} = (\alpha a_1, \alpha a_2, \alpha a_3)$

NOTE: \mathbf{a} and \mathbf{b} are *parallel* iff $\mathbf{a} = \lambda \mathbf{b}$ for some number λ .

D. **Magnitude (Norm):** $\|\mathbf{a}\| = \sqrt{a_1^2 + a_2^2 + a_3^2}$; \mathbf{a} is a *unit vector* if $\|\mathbf{a}\| = 1$. If $\mathbf{a} \neq \mathbf{0}$, then $\mathbf{u}_\mathbf{a} = \frac{\mathbf{a}}{\|\mathbf{a}\|}$ is a unit vector in the direction of \mathbf{a} .

(i) $\|\mathbf{a}\| \geq 0$; $\|\mathbf{a}\| = 0$ iff $\mathbf{a} = \mathbf{0}$.

(ii) $\|\alpha \mathbf{a}\| = |\alpha| \|\mathbf{a}\|$.

(iii) $\|\mathbf{a} + \mathbf{b}\| \leq \|\mathbf{a}\| + \|\mathbf{b}\|$.

E. **Unit Coordinate Vectors:** $\mathbf{i} = (1, 0, 0)$, $\mathbf{j} = (0, 1, 0)$, $\mathbf{k} = (0, 0, 1)$

F. **$\mathbf{i}, \mathbf{j}, \mathbf{k}$ -Representation:** $\mathbf{a} = (a_1, a_2, a_3) = a_1 \mathbf{i} + a_2 \mathbf{j} + a_3 \mathbf{k}$.

3. Dot Product: $\mathbf{a} \cdot \mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$.

The dot product of two n -dimensional vectors is defined similarly.

A. Properties:

(i) $\mathbf{a} \cdot \mathbf{a} = \|\mathbf{a}\|^2$

(ii) $\mathbf{a} \cdot \mathbf{0} = 0$

(iii) $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$

(iv) $(\alpha\mathbf{a}) \cdot (\beta\mathbf{b}) = \alpha\beta(\mathbf{a} \cdot \mathbf{b})$

(v) $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$

B. Geometric Interpretation: $\mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\| \|\mathbf{b}\| \cos \theta$ where θ is the angle between \mathbf{a} and \mathbf{b} , $0 \leq \theta \leq \pi$. \mathbf{a} is *perpendicular* to \mathbf{b} ($\mathbf{a} \perp \mathbf{b}$) iff $\mathbf{a} \cdot \mathbf{b} = 0$.

C. Component & Projection of \mathbf{a} on \mathbf{b} :

$$\text{comp}_{\mathbf{b}} \mathbf{a} = \frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{b}\|} = \mathbf{a} \cdot \mathbf{u}_{\mathbf{b}} \quad \text{proj}_{\mathbf{b}} \mathbf{a} = \frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{b}\|} \cdot \frac{\mathbf{b}}{\|\mathbf{b}\|} = (\mathbf{a} \cdot \mathbf{u}_{\mathbf{b}}) \mathbf{u}_{\mathbf{b}}$$

D. Direction Angles; Direction Cosines: $\mathbf{a} \cdot \mathbf{i} = \|\mathbf{a}\| \cos \alpha$, $\mathbf{a} \cdot \mathbf{j} = \|\mathbf{a}\| \cos \beta$, $\mathbf{a} \cdot \mathbf{k} = \|\mathbf{a}\| \cos \gamma$. The angles α , β , γ are called the *direction angles* of \mathbf{a} ; $\cos \alpha$, $\cos \beta$, $\cos \gamma$ are called the *direction cosines* of \mathbf{a} .

4. Cross Product: This product is restricted to vectors in 3-space. Let \mathbf{a} and \mathbf{b} be vectors in 3-space such that $\mathbf{a} \neq \lambda\mathbf{b}$. The *cross product* of \mathbf{a} and \mathbf{b} , denoted by $\mathbf{a} \times \mathbf{b}$ is the vector defined as follows:

(1) $\mathbf{a} \times \mathbf{b}$ is perpendicular to the plane determined by \mathbf{a} and \mathbf{b} .

(2) \mathbf{a} , \mathbf{b} and $\mathbf{a} \times \mathbf{b}$ (in this order) form a right-handed triple.

(3) $\|\mathbf{a} \times \mathbf{b}\| = \|\mathbf{a}\| \|\mathbf{b}\| \sin \theta$ where θ is the angle between \mathbf{a} and \mathbf{b} .

If $\mathbf{a} = \lambda\mathbf{b}$ for some number λ ; that is, if \mathbf{a} and \mathbf{b} are parallel, then $\mathbf{a} \times \mathbf{b} = \mathbf{0}$.

A. Properties:

(i) $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$.

(ii) $(\alpha\mathbf{a}) \times (\beta\mathbf{b}) = \alpha\beta(\mathbf{a} \times \mathbf{b})$.

(iii) $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$.

B. Components of $\mathbf{a} \times \mathbf{b}$: Let $\mathbf{a} = (a_1, a_2, a_3)$, $\mathbf{b} = (b_1, b_2, b_3)$

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = (a_2b_3 - a_3b_2)\mathbf{i} - (a_1b_3 - a_3b_1)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}$$

C. **Triple Scalar Product:** Let $\mathbf{a} = (a_1, a_2, a_3)$, $\mathbf{b} = (b_1, b_2, b_3)$, $\mathbf{c} = (c_1, c_2, c_3)$. Then

$$(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

The volume of the parallelepiped having \mathbf{a} , \mathbf{b} and \mathbf{c} as sides is given by:

$$|(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}|$$

5. Lines: There is one and only one line ℓ passing through a given point $P_0 : (x_0, y_0, z_0)$ parallel to a given vector $\mathbf{d} = (d_1, d_2, d_3)$. The vector \mathbf{d} is called a *direction vector* for ℓ ; the numbers d_1, d_2, d_3 are *direction numbers* for ℓ .

A. **Equations for ℓ :**

Vector Equation of ℓ :

$$\mathbf{r}(t) = \mathbf{r}_0 + t\mathbf{d} = (x_0 + td_1)\mathbf{i} + (y_0 + td_2)\mathbf{j} + (z_0 + td_3)\mathbf{k}$$

Scalar Parametric Equations of ℓ :

$$x(t) = x_0 + td_1, \quad y(t) = y_0 + td_2, \quad z(t) = z_0 + td_3$$

Symmetric Equations of ℓ :

$$\frac{x - x_0}{d_1} = \frac{y - y_0}{d_2} = \frac{z - z_0}{d_3}$$

B. **Two Lines in Space:** Let ℓ and \mathcal{L} be two lines in space, and let \mathbf{d} and \mathbf{D} be corresponding direction vectors. ℓ and \mathcal{L} are either parallel or coincident if $\mathbf{d} = \lambda\mathbf{D}$; ℓ and \mathcal{L} intersect in a point or are *skew* if $\mathbf{d} \neq \lambda\mathbf{D}$.

C. **Angle ϕ between ℓ and \mathcal{L} :** $\cos \phi = \frac{|\mathbf{d} \cdot \mathbf{D}|}{\|\mathbf{d}\| \|\mathbf{D}\|}$

D. **Distance From a Point $P_1 : (x_1, y_1, z_1)$ to the Line ℓ :**

$$d(P_1, \ell) = \frac{\|\overrightarrow{P_0P_1} \times \mathbf{d}\|}{\|\mathbf{d}\|}$$

6. Planes: There is one and only one plane \mathcal{P} passing through a given point $P_0 : (x_0, y_0, z_0)$ perpendicular to a given vector $\mathbf{N} = A\mathbf{i} + B\mathbf{j} + C\mathbf{k}$. The vector \mathbf{N} is called a *normal vector* to the plane \mathcal{P} .

A. Equations for \mathcal{P} :

”Standard Form:” $A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$

Removing the parentheses yields the equation

$$Ax + By + Cz + D = 0, \quad D = -Ax_0 - By_0 - Cz_0$$

which can also be written in the form

$$Ax + By + Cz = E.$$

B. Two Planes in Space: Let \mathcal{P}_1 and \mathcal{P}_2 be two planes in space, and let \mathbf{N}_1 and \mathbf{N}_2 be corresponding normal vectors. \mathcal{P}_1 and \mathcal{P}_2 are either parallel or coincident if $\mathbf{N}_1 = \lambda\mathbf{N}_2$; \mathcal{P}_1 and \mathcal{P}_2 intersect in a line if $\mathbf{N}_1 \neq \lambda\mathbf{N}_2$.

C. Angle θ between \mathcal{P}_1 and \mathcal{P}_2 : $\cos \theta = \frac{|\mathbf{N}_1 \cdot \mathbf{N}_2|}{\|\mathbf{N}_1\| \|\mathbf{N}_2\|}$

C. Distance From a Point $P_1 : (x_1, y_1, z_1)$ to the Plane \mathcal{P} :

$$d(P_1, \mathcal{P}) = \frac{\|Ax_1 + By_1 + Cz_1 + D\|}{\sqrt{A^2 + B^2 + C^2}}$$