CHAPTER 12

SECTION 12.1

3.

length $\overline{AB}$: $5\sqrt{2}$

midpoint: $\left(2, -\frac{1}{2}, \frac{5}{2}\right)$

4.

length $\overline{AB}$: 9

midpoint: $(1, \frac{3}{2}, 3)$

5. $z = -2$

7. $y = 1$

9. $x = 3$

12. $(x-1)^2 + y^2 + (z+2)^2 = 16$

13. $(x-2)^2 + (y-4)^2 + (z+4)^2 = 36$

15. $(x-3)^2 + (y-2)^2 + (z-2)^2 = 13$

19.

$$x^2 + y^2 + z^2 + 4x - 8y - 2z + 5 = 0$$

$$x^2 + 4x + 4 + y^2 - 8y + 16 + z^2 - 2z + 1 = -5 + 4 + 16 + 1$$

$$(x+2)^2 + (y-4)^2 + (z-1)^2 = 16$$

center: $(-2, 4, 1)$, radius: 4

20. Rewrite as $x^2 - 4x + 4 + y^2 + z^2 - 2z + 1 = -1 + 4 + 1 = 4$

$\Longrightarrow$ $(x-2)^2 + y^2 + (z-1)^2 = 4$ center $(2, 0, 1)$; radius 2

35. Each such sphere has an equation of the form

$$(x-a)^2 + (y-a)^2 + (z-a)^2 = a^2.$$

Substituting $x = 5$, $y = 1$, $z = 4$ we get

$$(5-a)^2 + (1-a)^2 + (4-a)^2 = a^2.$$

This reduces to $a^2 - 10a + 21 = 0$ and gives $a = 3$ or $a = 7$. The equations are:

$$(x-3)^2 + (y-3)^2 + (z-3)^2 = 9; \quad (x-7)^2 + (y-7)^2 + (z-7)^2 = 49$$

37. Not a sphere; this equation is equivalent to:

$$(x-2)^2 + (y+2)^2 + (z+3)^2 = -3$$

which has no (real) solutions.

38. $x^2 + y^2 + z^2 + Ax + By + Cz + D = 0$

$\Longrightarrow \left(x + \frac{A}{2}\right)^2 + \left(y + \frac{B}{2}\right)^2 + \left(z + \frac{C}{2}\right)^2 = \frac{A^2}{4} + \frac{B^2}{4} + \frac{C^2}{4} - D,$

so you get a sphere if $\frac{A^2}{4} + \frac{B^2}{4} + \frac{C^2}{4} - D > 0.$

41. The sphere of radius 2 centered at the origin, together with its interior.

43. A rectangular box in the first octant with sides on the coordinate planes and dimensions $1 \times 2 \times 3$, together with its interior.

45. A circular cylinder with base the circle $x^2 + y^2 = 4$ and height 4, together with its interior.

46. $x^2 + y^2 + z^2 = 4$ and $x^2 + y^2 + z^2 = 9$ are concentric spheres; Ω is the region between the two spheres.

Section 12.3

1. $\overrightarrow{PQ} = (3, 4, -2)$; $\quad \|\overrightarrow{PQ}\| = \sqrt{29}$

4. $\overrightarrow{PQ} = (4, 3, -8)$; $\quad \|\overrightarrow{PQ}\| = \sqrt{89}$

7. $-2\mathbf{a} + \mathbf{b} - \mathbf{c} = [-2(\mathbf{a} - \mathbf{b})] - \mathbf{c} = (1 + 4, 4 - 2, -7 - 1) = (5, 2, -8)$

8. $\mathbf{a} + 3\mathbf{b} - 2\mathbf{c} = (1, -2, 3) + 3(3, 0, -1) - 2(-4, 2, 1) = (18, -6, -2).$

10. $3\mathbf{i} + 5\mathbf{j} + \mathbf{k}$

11. $-3\mathbf{i} - \mathbf{j} + 8\mathbf{k}$

15. 3

17. $\sqrt{6}$

19. (a) $\mathbf{a}$, $\mathbf{c}$, and $\mathbf{d}$ since $\mathbf{a} = \frac{1}{3}\mathbf{c} = -\frac{1}{2}\mathbf{d}$

(b) $\mathbf{a}$ and $\mathbf{c}$ since $\mathbf{a} = \frac{1}{3}\mathbf{c}$

(c) $\mathbf{a}$ and $\mathbf{c}$ both have direction opposite to $\mathbf{d}$

21. $\|\mathbf{a}\| = 5$; $\quad \dfrac{\mathbf{a}}{\|\mathbf{a}\|} = \left(\dfrac{3}{5}, -\dfrac{4}{5}, 0\right)$

23. $\|\mathbf{a}\| = 3$; $\quad \dfrac{\mathbf{a}}{\|\mathbf{a}\|} = \dfrac{1}{3}\mathbf{i} - \dfrac{2}{3}\mathbf{j} + \dfrac{2}{3}\mathbf{k}$

24. $\left(\dfrac{2}{3}, \dfrac{1}{3}, \dfrac{2}{3}\right)$

25. $\|\mathbf{a}\| = \sqrt{14}$; $\quad -\dfrac{\mathbf{a}}{\|\mathbf{a}\|} = \dfrac{1}{\sqrt{14}}\mathbf{i} - \dfrac{3}{\sqrt{14}}\mathbf{j} - \dfrac{2}{\sqrt{14}}\mathbf{k}$

28. (a) $6\mathbf{i} + 3\mathbf{j} + 12\mathbf{k}$

(b) $A(1, 1, 1) + B(-1, 3, 2) + C(-3, 0, 1) = (4, -1, 1).$

Solve simultaneously to get $A = \frac{26}{7}$, $\quad B = -\frac{11}{7}$, $\quad C = \frac{3}{7}$

30. $\alpha = -12$

31. $\|3\mathbf{i} + \mathbf{j}\| = \|\alpha\mathbf{j} - \mathbf{k}\| \implies 10 = \alpha^2 + 1$ so $\alpha = \pm 3$

33.

$$\|\alpha\mathbf{i} + (\alpha - 1)\,\mathbf{j} + (\alpha + 1)\,\mathbf{k}\| = 2 \implies \alpha^2 + (\alpha - 1)^2 + (\alpha + 1)^2 = 4$$

$$\implies 3\alpha^2 = 2 \quad \text{so} \quad \alpha = \pm\tfrac{1}{3}\sqrt{6}$$

34. $\dfrac{2}{\sqrt{6}}(\mathbf{i} + 2\mathbf{j} - \mathbf{k}) = \dfrac{\sqrt{6}}{3}\,(\mathbf{i} + 2\mathbf{j} - \mathbf{k})$

41. (a) $\|\mathbf{r} - \mathbf{a}\| = 3$ where $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$

(b) $\|\mathbf{r}\| \le 2$ (c) $\|\mathbf{r}-\mathbf{a}\| \le 1$ where $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$

Section 12.4

1. $\mathbf{a}\cdot\mathbf{b} = (2)(-2) + (-3)(0) + (1)(3) = -1$ **3.** $\mathbf{a}\cdot\mathbf{b} = (2)(1) + (-4)(1/2) + (0)(0) = 0$

5. $\mathbf{a}\cdot\mathbf{b} = (2)(1) + (1)(1) - (2)(2) = -1$

8. $\mathbf{a}\cdot(\mathbf{a}-\mathbf{b}) + \mathbf{b}\cdot(\mathbf{b}+\mathbf{a}) = \mathbf{a}\cdot\mathbf{a} - \mathbf{a}\cdot\mathbf{b} + \mathbf{b}\cdot\mathbf{b} + \mathbf{b}\cdot\mathbf{a} = \|\mathbf{a}\|^2 + \|\mathbf{b}\|^2$

9. $(\mathbf{a}-\mathbf{b})\cdot\mathbf{c} + \mathbf{b}\cdot(\mathbf{c}+\mathbf{a}) = \mathbf{a}\cdot\mathbf{c} - \mathbf{b}\cdot\mathbf{c} + \mathbf{b}\cdot\mathbf{c} + \mathbf{b}\cdot\mathbf{a} = \mathbf{a}\cdot(\mathbf{b}+\mathbf{c})$

11. (a) $\mathbf{a}\cdot\mathbf{b} = (2)(3) + (1)(-1) + (0)(2) = 5$

$\mathbf{a}\cdot\mathbf{c} = (2)(4) + (1)(0) + (0)(3) = 8$

$\mathbf{b}\cdot\mathbf{c} = (3)(4) + (-1)(0) + (2)(3) = 18$

(b) $\|\mathbf{a}\| = \sqrt{5}$, $\|\mathbf{b}\| = \sqrt{14}$, $\|\mathbf{c}\| = 5$. Then,

$$\cos\sphericalangle(\mathbf{a},\mathbf{b}) = \frac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{a}\|\,\|\mathbf{b}\|} = \frac{5}{(\sqrt{5})\,(\sqrt{14})} = \frac{1}{14}\sqrt{70},$$

$$\cos\sphericalangle(\mathbf{a},\mathbf{c}) = \frac{8}{(\sqrt{5})\,(5)} = \frac{8}{25}\sqrt{5},$$

$$\cos\sphericalangle(\mathbf{b},\mathbf{c}) = \frac{18}{(\sqrt{14})\,(5)} = \frac{9}{35}\sqrt{14}.$$

(c) $\mathbf{u_b} = \dfrac{1}{\sqrt{14}}(3\mathbf{i} - \mathbf{j} + 2\mathbf{k})$, $\text{comp}_\mathbf{b}\,\mathbf{a} = \mathbf{a}\cdot\mathbf{u_b} = \dfrac{1}{\sqrt{14}}(6-1) = \dfrac{5}{14}\sqrt{14}$,

$\mathbf{u_c} = \dfrac{1}{5}(4\mathbf{i} + 3\mathbf{k})$, $\text{comp}_\mathbf{c}\,\mathbf{a} = \mathbf{a}\cdot\mathbf{u_c} = \dfrac{8}{5}$

(d) $\mathbf{proj_b}\,\mathbf{a} = (\text{comp}_\mathbf{b}\,\mathbf{a})\,\mathbf{u_b} = \dfrac{5}{14}(3\mathbf{i} - \mathbf{j} + 2\mathbf{k})$, $\mathbf{proj_c}\,\mathbf{a} = (\text{comp}_\mathbf{c}\,\mathbf{a})\,\mathbf{u_c} = \dfrac{8}{25}(4\mathbf{i} + 3\mathbf{k})$

13. $\mathbf{u} = \cos\dfrac{\pi}{3}\,\mathbf{i} + \cos\dfrac{\pi}{4}\,\mathbf{j} + \cos\dfrac{2\pi}{3}\,\mathbf{k} = \dfrac{1}{2}\mathbf{i} + \dfrac{1}{2}\sqrt{2}\,\mathbf{j} - \dfrac{1}{2}\mathbf{k}$

14. $\mathbf{v} = 2(\cos\dfrac{\pi}{4}\mathbf{i} + \cos\dfrac{\pi}{4}\mathbf{j} + \cos\dfrac{\pi}{2}\mathbf{k}) = \sqrt{2}\mathbf{i} + \sqrt{2}\mathbf{j}.$

15. $\cos\theta = \dfrac{(3\mathbf{i} - \mathbf{j} - 2\mathbf{k})\cdot(\mathbf{i} + 2\mathbf{j} - 3\mathbf{k})}{\|3\mathbf{i} - \mathbf{j} - 2\mathbf{k}\|\,\|\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}\|} = \dfrac{7}{\sqrt{14}\,\sqrt{14}} = \dfrac{1}{2}$, $\theta = \dfrac{\pi}{3}$

17. Since $\|\mathbf{i} - \mathbf{j} + \sqrt{2}\,\mathbf{k}\| = 2$, we have $\cos\alpha = \frac{1}{2}$, $\cos\beta = -\frac{1}{2}$, $\cos\gamma = \frac{1}{2}\sqrt{2}$.

The direction angles are $\frac{1}{3}\pi$, $\frac{2}{3}\pi$, $\frac{1}{4}\pi$.

19. $\theta = \cos^{-1}\dfrac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{a}\|\|\mathbf{b}\|} = \cos^{-1}\left(\dfrac{-1}{\sqrt{231}}\right) \cong 2.2$ radians or $126.3°$

21. $\theta = \cos^{-1}\dfrac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{a}\|\|\mathbf{b}\|} = \cos^{-1}\left(\dfrac{-13}{5\sqrt{10}}\right) \cong 2.54$ radians or $145.3°$

27. $\|\mathbf{a}\| = \sqrt{3^2 + (12)^2 + 4^2} = 13;$ $\cos\alpha = \dfrac{3}{13}, \quad \cos\beta = \dfrac{12}{13} \quad \cos\gamma = \dfrac{4}{13}$

$\alpha \cong 76.7^\circ \quad \beta \cong 22.6^\circ, \quad \gamma \cong 72.1^\circ$

29. $2\mathbf{i} + 5\mathbf{j} + 2x\mathbf{k} \perp 6\mathbf{i}|4\mathbf{j} - x\mathbf{k} \implies 12 + 20 - 2x^2 = 0 \implies x^2 = 16 \implies x = \pm 4$

31. $\cos\dfrac{\pi}{3} = \dfrac{\mathbf{c}\cdot\mathbf{d}}{\|\mathbf{c}\|\ \|\mathbf{d}\|}, \quad \dfrac{1}{2} = \dfrac{2x+1}{x^2+2}, \quad x^2 = 4x; \qquad x = 0, \quad x = 4$

32. $(\mathbf{i} + x\mathbf{j} + \mathbf{k}) \cdot (2\mathbf{i} - \mathbf{j} + y\mathbf{k}) = 0 \implies 2 - x + y = 0$

$1 + x^2 + 1 = 4 + 1 + y^2 \implies x^2 - y^2 = 3 \implies x = \dfrac{7}{4}, \quad y = -\dfrac{1}{4}$

37. Set $\mathbf{u} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$. The relations

$$(a\mathbf{i} + b\mathbf{j} + c\mathbf{k}) \cdot (\mathbf{i} + 2\mathbf{j} + \mathbf{k}) = 0 \quad \text{and} \quad (a\mathbf{i} + b\mathbf{j} + c\mathbf{k}) \cdot (3\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}) = 0$$

give

$$a + 2b + c = 0 \qquad 3a - 4b + 2c = 0$$

so that $b = \frac{1}{8}a$ and $c = -\frac{5}{4}a$.

Then, since $\mathbf{u}$ is a unit vector,

$$a^2 + b^2 + c^2 = 1, \quad a^2 + \left(\frac{a}{8}\right)^2 + \left(\frac{-5a}{4}\right)^2 = 1, \quad \frac{165}{64}a^2 = 1.$$

Thus, $a = \pm\dfrac{8}{165}\sqrt{165}$ and $\mathbf{u} = \pm\dfrac{\sqrt{165}}{165}(8\mathbf{i} + \mathbf{j} - 10\mathbf{k})$.

38. $\pm\mathbf{k}, \quad \pm\dfrac{\sqrt{13}}{13}(3\mathbf{i} - 2\mathbf{j})$

39. We take $\mathbf{u} = \mathbf{i}$ as an edge and $\mathbf{v} = \mathbf{i} + \mathbf{j} + \mathbf{k}$ as a diagonal of a cube. Then,

$\cos\theta = \dfrac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\ \|\mathbf{v}\|} = \dfrac{1}{3}\sqrt{3}, \qquad \theta = \cos^{-1}\left(\tfrac{1}{3}\sqrt{3}\right) \cong 0.96$ radians.

44. (a) $\|\mathbf{a} + \mathbf{b}\|^2 = \|\mathbf{a}\|^2 + \|\mathbf{b}\|^2 + 2\mathbf{a}\cdot\mathbf{b} = \|\mathbf{a}\|^2 + \|\mathbf{b}\|^2 \implies \mathbf{a} \perp \mathbf{b}$.

(b) $\|\mathbf{a} - \mathbf{b}\|^2 = \|\mathbf{a}\|^2 + \|\mathbf{b}\|^2 - 2\mathbf{a}\cdot\mathbf{b} = \|\mathbf{a}\|^2 + \|\mathbf{b}\|^2 \implies \mathbf{a} \perp \mathbf{b}$.

45. (a) $\|\mathbf{a} + \mathbf{b}\|^2 - \|\mathbf{a} - \mathbf{b}\|^2 = (\mathbf{a} + \mathbf{b}) \cdot (\mathbf{a} + \mathbf{b}) - (\mathbf{a} - \mathbf{b}) \cdot (\mathbf{a} - \mathbf{b})$

$$= [(\mathbf{a}\cdot\mathbf{a}) + 2(\mathbf{a}\cdot\mathbf{b}) + (\mathbf{b}\cdot\mathbf{b})] - [(\mathbf{a}\cdot\mathbf{a}) - 2(\mathbf{a}\cdot\mathbf{b}) + (\mathbf{b}\cdot\mathbf{b})] = 4(\mathbf{a}\cdot\mathbf{b})$$

(b) The following statements are equivalent:

$$\mathbf{a} \perp \mathbf{b}, \quad \mathbf{a}\cdot\mathbf{b} = 0, \quad \|\mathbf{a} + \mathbf{b}\|^2 - \|\mathbf{a} - \mathbf{b}\|^2 = 0, \quad \|\mathbf{a} + \mathbf{b}\| = \|\mathbf{a} - \mathbf{b}\|.$$

(c) By (b), the relation $\|\mathbf{a} + \mathbf{b}\| = \|\mathbf{a} - \mathbf{b}\|$ gives $\mathbf{a} \perp \mathbf{b}$. The relation $\mathbf{a} + \mathbf{b} \perp \mathbf{a} - \mathbf{b}$ gives

$$0 = (\mathbf{a} + \mathbf{b}) \cdot (\mathbf{a} - \mathbf{b}) = \|\mathbf{a}\|^2 - \|\mathbf{b}\|^2 \quad \text{and thus} \quad \|\mathbf{a}\| = \|\mathbf{b}\|.$$

The parallelogram is a square since it has two adjacent sides of equal length and these meet at right angles.

46. $|\mathbf{a}\cdot\mathbf{b}| = \|\mathbf{a}\|\|\mathbf{b}\||\cos\theta| = \|\mathbf{a}\|\|\mathbf{b}\|$ iff $\theta = 0$ or $\theta = \pi$

Section 12.5

4. $\mathbf{j}\times(2\mathbf{i}-\mathbf{k})=\mathbf{j}\times 2\mathbf{i}-\mathbf{j}\times\mathbf{k}=-2\mathbf{k}-\mathbf{i}=-\mathbf{i}-2\mathbf{k}$

5. $(2\mathbf{j}-\mathbf{k})\times(\mathbf{i}-3\mathbf{j})=[2\mathbf{j}\times(\mathbf{i}-3\mathbf{j})]-[\mathbf{k}\times(\mathbf{i}-3\mathbf{j})]=(-2\mathbf{k})-(\mathbf{j}+3\mathbf{i})=-3\mathbf{i}-\mathbf{j}-2\mathbf{k}$

or

$$(2\mathbf{j}-\mathbf{k})\times(\mathbf{i}-3\mathbf{j})=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\0&2&-1\\1&-3&0\end{vmatrix}=\mathbf{i}\begin{vmatrix}2&-1\\-3&0\end{vmatrix}-\mathbf{j}\begin{vmatrix}0&-1\\1&-3\end{vmatrix}+\mathbf{k}\begin{vmatrix}0&2\\1&-3\end{vmatrix}=-3\mathbf{i}-\mathbf{j}-2\mathbf{k}$$

7. $\mathbf{j}\cdot(\mathbf{i}\times\mathbf{k})=\mathbf{j}\cdot(-\mathbf{j})=-1$

8. $(\mathbf{j}\times\mathbf{i})\cdot(\mathbf{i}\times\mathbf{k})=(-\mathbf{k})\cdot(-\mathbf{j})=\mathbf{0}$

14. $(3\mathbf{i}-2\mathbf{j}+\mathbf{k})\times(\mathbf{i}-\mathbf{j}+\mathbf{k})=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\3&-2&1\\1&-1&1\end{vmatrix}=-\mathbf{i}-2\mathbf{j}-\mathbf{k}$

15. $(\mathbf{i}+\mathbf{j}+\mathbf{k})\times(2\,\mathbf{i}+\mathbf{k})=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\1&1&1\\2&0&1\end{vmatrix}=[(1)(1)-(1)(0)]\,\mathbf{i}-[(1)(1)-(1)(2)]\,\mathbf{j}+[(1)(0)-(1)(2)]\,\mathbf{k}$

$$=\mathbf{i}+\mathbf{j}-2\,\mathbf{k}$$

17. $[2\mathbf{i}+\mathbf{j}]\cdot[(\mathbf{i}-3\mathbf{j}+\mathbf{k})\times(4\mathbf{i}+\mathbf{k})]=\begin{vmatrix}1&-3&1\\4&0&1\\2&1&0\end{vmatrix}=$

$$[(0)(0)-(1)(1)]-(-3)[(4)(0)-(1)(2)]+[(4)(1)-(0)(2)]=-3$$

18. $[(-2\mathbf{i}+\mathbf{j}-3\mathbf{k})\times\mathbf{i}]\times[\mathbf{i}+\mathbf{j}]=(-3\mathbf{j}-\mathbf{k})\times(\mathbf{i}+\mathbf{j})=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\0&-3&-1\\1&1&0\end{vmatrix}=\mathbf{i}-\mathbf{j}+3\mathbf{k}$

21. $\mathbf{a}\times\mathbf{b}=\begin{vmatrix}1&-3&1\\4&0&1\\2&1&0\end{vmatrix}=3\mathbf{i}-3\mathbf{j}-6\mathbf{k}$

$$\frac{\mathbf{a}\times\mathbf{b}}{\|\mathbf{a}\times\mathbf{b}\|}=\frac{1}{\sqrt{6}}\mathbf{i}-\frac{1}{\sqrt{6}}\mathbf{j}-\frac{2}{\sqrt{6}}\mathbf{k};\quad \frac{\mathbf{b}\times\mathbf{a}}{\|\mathbf{b}\times\mathbf{a}\|}=-\frac{1}{\sqrt{6}}\mathbf{i}+\frac{1}{\sqrt{6}}\mathbf{j}+\frac{2}{\sqrt{6}}\mathbf{k}$$

23. Set $\mathbf{a}=\overrightarrow{PQ}=-\mathbf{i}+2\mathbf{k}$ and $\mathbf{b}=\overrightarrow{PR}=2\mathbf{i}-\mathbf{k}$. Then

$$\mathbf{N}=\overrightarrow{PQ}\times\overrightarrow{PR}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\-1&0&2\\2&0&-1\end{vmatrix}=3\mathbf{j}$$

and $A=\frac{1}{2}\,\|\,\mathbf{a}\times\mathbf{b}\,\|=\frac{1}{2}\,\|\,3\mathbf{j}\,\|=\frac{3}{2}$.

26. $\mathbf{N}=\overrightarrow{PQ}\times\overrightarrow{PR}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\2&2&-4\\-5&1&2\end{vmatrix}=8\mathbf{i}+16\mathbf{j}+12\mathbf{k}$

Area$=\frac{1}{2}\|\mathbf{N}\|=2\sqrt{29}$

28. $V = (\mathbf{i} - 3\mathbf{j} + \mathbf{k}) \times (2\mathbf{j} - \mathbf{k})] \cdot (\mathbf{i} + \mathbf{j} - 2\mathbf{k}) = \begin{vmatrix} 1 & -3 & 1 \\ 0 & 2 & -1 \\ 1 & 1 & -2 \end{vmatrix} = -2; \qquad V = |-2| = 2$

29. $V = \left|\overrightarrow{OP} \cdot \left(\overrightarrow{OQ} \times \overrightarrow{OR}\right)\right| = \left|\begin{vmatrix} 1 & 2 & 3 \\ 1 & 1 & 2 \\ 2 & 1 & 1 \end{vmatrix}\right| = 2$

34. $\mathbf{a} \times \mathbf{b} = (a_1 b_2 - b_1 a_2)\mathbf{k}$

35.
$$\begin{aligned}(\alpha\mathbf{a} + \beta\mathbf{b}) \times (\gamma\mathbf{a} + \delta\mathbf{b}) &= (\alpha\mathbf{a} \times \delta\mathbf{b}) + (\beta\mathbf{b} \times \gamma\mathbf{a}) \\ &= \alpha\delta(\mathbf{a} \times \mathbf{b}) - \beta\gamma(\mathbf{a} \times \mathbf{b}) \\ &= (\alpha\delta - \beta\gamma)(\mathbf{a} \times \mathbf{b}) = \begin{vmatrix} \alpha & \beta \\ \gamma & \delta \end{vmatrix}(\mathbf{a} \times \mathbf{b})\end{aligned}$$

38. Since $\mathbf{a} \times \mathbf{b} \perp \mathbf{b}$, $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{b} = \mathbf{0}$.

41. $\mathbf{a} \cdot \mathbf{b} = \mathbf{a} \cdot \mathbf{c} \implies \mathbf{a} \cdot (\mathbf{b} - \mathbf{c}) = 0$; $\mathbf{a}$ is perpendicular to $\mathbf{b} - \mathbf{c}$.

$\mathbf{a} \times \mathbf{b} = \mathbf{a} \times \mathbf{c} \implies \mathbf{a} \times (\mathbf{b} - \mathbf{c}) = \mathbf{0}$; $\mathbf{a}$ is parallel to $\mathbf{b} - \mathbf{c}$.

Since $\mathbf{a} \neq \mathbf{0}$ it follows that $\mathbf{b} - \mathbf{c} = \mathbf{0}$ or $\mathbf{b} = \mathbf{c}$.

45. Suppose $\mathbf{a} \neq \mathbf{0}$. Then

$$\mathbf{a} \cdot \mathbf{b} = 0 \implies \mathbf{b} \perp \mathbf{a}; \quad \mathbf{a} \times \mathbf{b} = \mathbf{0} \implies \mathbf{b} \| \mathbf{a}$$

Thus $\mathbf{b}$ is simultaneously perpendicular to, and parallel to $\mathbf{a}$. It follows that $\mathbf{b} = \mathbf{0}$.

48. (a) $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$: makes sense – this is the dot product of two vectors.

(b) $\mathbf{a} \times (\mathbf{b} \cdot \mathbf{c})$: does not make sense – this is the cross product of a vector with a number.

(c) $\mathbf{a} \cdot (\mathbf{b} \cdot \mathbf{c})$: does not make sense – this is the dot product of a vector with a number.

(d) $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$: makes sense – this is the cross product of two vectors.

Section 12.6

1. P (when $t = 0$) and Q (when $t = -1$)

2. l_1, l_3 and l_4 are parallel.

3. Take $\mathbf{r}_0 = \overrightarrow{OP} = 3\mathbf{i} + \mathbf{j}$ and $\mathbf{d} = \mathbf{k}$. Then, $\mathbf{r}(t) = (3\mathbf{i} + \mathbf{j}) + t\mathbf{k}$.

4. $\mathbf{r}(t) = \mathbf{i} - \mathbf{j} + 2\mathbf{k} + t(3\mathbf{i} - \mathbf{j} + \mathbf{k})$

7. $\overrightarrow{PQ} = \mathbf{i} - \mathbf{j} + \mathbf{k}$ so direction numbers are $1, -1, 1$. Using P as a point on the line, we have

$$x(t) = 1 + t, \quad y(t) = -t, \quad z(t) = 3 + t.$$

10. $x(t) = 1 + t, \quad y(t) = 4, \quad z(t) = -3$

11. Since the line $2(x + 1) = 4(y - 3) = z$ can be written

$$\frac{x+1}{2} = \frac{y-3}{1} = \frac{z}{4},$$

it has direction numbers $2, 1, 4$. The line through $P(-1, 2, -3)$ with direction vector $2\mathbf{i} + \mathbf{j} + 4\mathbf{k}$ can be parameterized

$$\mathbf{r}(t) = (-\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}) + t(2\mathbf{i} + \mathbf{j} + 4\mathbf{k}).$$

13. $\mathbf{r}(t) = (3\mathbf{i} + \mathbf{j} + 5\mathbf{k}) + t(\mathbf{i} - \mathbf{j} + 2\mathbf{k}) = (3 + t)\mathbf{i} + (1 - t)\mathbf{j} + (5 + 2t)\mathbf{k}$

$\mathbf{R}(u) = (\mathbf{i} + 4\mathbf{j} + 2\mathbf{k}) + u(\mathbf{j} + \mathbf{k}) = \mathbf{i} + (4 + u)\mathbf{j} + (2 + u)\mathbf{k}$

$\mathbf{d} = \mathbf{i} - \mathbf{j} + 2\mathbf{k}$ is a direction vector for l_1; $\mathbf{D} = \mathbf{j} + \mathbf{k}$ is a direction vector for l_2. Since $\mathbf{d}$ is not a multiple of $\mathbf{D}$, the lines either intersect or are skew. Setting $\mathbf{r}(t) = \mathbf{R}(u)$ we get the system of equations:

$$3 + t = 1, \quad 1 - t = 4 + u, \quad 5 + 2t = 2 + u$$

This system has the solution $t = -2$, $u = -1$. The point of intersection is: $(1, 3, 1)$.

17. $\mathbf{d} = -6\mathbf{i} + 9\mathbf{j} - 3\mathbf{k}$ is a direction vector for l_1; $\mathbf{D} = 2\mathbf{i} - 3\mathbf{j} + \mathbf{k}$ is a direction vector for l_2. Since $\mathbf{d} = 3\mathbf{D}$, we conclude that l_1 and l_2 are either parallel or coincident. The point $(1, 2, 0)$ lies on l_1 but does not line on l_2. Therefore, the lines are parallel.

18. $\mathbf{d} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$ is a direction vector for l_1; $\mathbf{D} = 3\mathbf{i} + 2\mathbf{j} + 1\mathbf{k}$ is a direction vector for l_2. Since $\mathbf{d}$ is not a multiple of $\mathbf{D}$, the lines either intersect or are skew. The system of equations

$$2 + t = 5 + 3u, \quad -1 + 2t = 1 + 2u, \quad 1 + 3t = 4 + u$$

does not have a solution. Therefore the lines are skew.

19. $\mathbf{d} = 2\mathbf{i} + 4\mathbf{j} + 3\mathbf{k}$ is a direction vector for l_1; $\mathbf{D} = \mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$ is a direction vector for l_2. Since $\mathbf{d}$ is not a multiple of $\mathbf{D}$, the lines either intersect or are skew. The system of equations

$$4 + 2t = 2 + u, \quad -5 + 4t = -1 + 3u, \quad 1 + 3t = 2u$$

does not have a solution. Therefore the lines are skew.

24. The lines meet at (x_0, y_0, z_0), and since $\mathbf{d} \cdot \mathbf{D} = \mathbf{0}$, they are perpendicular.

25. The lines are parallel.

26. Note that $\mathbf{r}(0) = \mathbf{r}_0$ and $\mathbf{r}(1) = \mathbf{r}_1$, so we need $0 \leq t \leq 1$

27. $\mathbf{r}(t) = (2\mathbf{i} + 7\mathbf{j} - \mathbf{k}) + t(2\mathbf{i} - 5\mathbf{j} + 4\mathbf{k}), \quad 0 \leq t \leq 1$

28. $-1 \leq t \leq 2$.

34. $d(P, l) = \dfrac{\|(\mathbf{j} + \mathbf{k}) \times (\mathbf{i} - 2\mathbf{j} - 2\mathbf{k})\|}{\|\mathbf{i} - 2\mathbf{j} - 2\mathbf{k}\|} = \dfrac{1}{3}\sqrt{2} \cong 0.47$

35. The line contains the point $P_0(1, 0, 2)$. Therefore

$$d(P, l) = \frac{\|(2\mathbf{j} + \mathbf{k}) \times (\mathbf{i} - 2\mathbf{j} + 3\mathbf{k})\|}{\|\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}\|} = \sqrt{\frac{69}{14}} \cong 2.22$$

Section 12.7

1. Q

3. Since $\mathbf{i}-4\mathbf{j}+3\mathbf{k}$ is normal to the plane, we have

$$(x-2)-4(y-3)+3(z-4)=0 \quad \text{and thus} \quad x-4y+3z-2=0.$$

6. $\mathbf{N}=4\mathbf{i}+2\mathbf{j}-7\mathbf{k}, \quad P(3,-1,5) \implies 4(x-3)+2(y+1)-7(z-5)=0$

7. The point $Q\,(0,0,-2)$ lies on the line l; and $\mathbf{d}=\mathbf{i}+\mathbf{j}+\mathbf{k}$ is a direction vector for l.

We want an equation for the plane which has the vector

$$\mathbf{N}=\overrightarrow{PQ}\times\mathbf{d}=(\mathbf{i}+3\mathbf{j}+3\mathbf{k})\times(\mathbf{i}+\mathbf{j}+\mathbf{k})$$

as a normal vector:

$$\mathbf{N}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 1 & 3 & 3\\ 1 & 1 & 1\end{vmatrix}=2\mathbf{j}-2\mathbf{k}$$

An equation for the plane is: $2(y-3)-2(z-1)=0$ or $y-z-2=0$

10. $\mathbf{N}=2\mathbf{i}-3\mathbf{j}+7\mathbf{k}$, unit normals: $\mathbf{u}_N=\pm\dfrac{1}{\sqrt{62}}(2\mathbf{i}-3\mathbf{j}+7\mathbf{k})$

13. Intercept form: $\dfrac{x}{15}+\dfrac{y}{12}-\dfrac{z}{10}=1$

x-intercept: $(15,0,0)$

y-intercept: $(0,12,0)$

z-intercept: $(0,0,-10)$

17. $\mathbf{u}_{\mathbf{N}_1}=\dfrac{\sqrt{3}}{3}(\mathbf{i}-\mathbf{j}+\mathbf{k}), \quad \mathbf{u}_{\mathbf{N}_2}=\dfrac{\sqrt{14}}{14}(2\mathbf{i}+\mathbf{j}+3\mathbf{k}), \quad \cos\theta=|\mathbf{u}_{\mathbf{N}_1}\cdot\mathbf{u}_{\mathbf{N}_2}|=\dfrac{2}{21}\sqrt{42}\cong 0.617.$

Therefore $\theta\cong 0.91$ radians.

18. $\cos\theta=\dfrac{|(4\mathbf{i}+4\mathbf{j}-2\mathbf{k})\cdot(2\mathbf{i}+\mathbf{j}+\mathbf{k})|}{\|4\mathbf{i}+4\mathbf{j}-2\mathbf{k}\|\|2\mathbf{i}+\mathbf{j}+\mathbf{k}\|}=\dfrac{10}{6\sqrt{6}}; \quad \theta\cong 0.82$ radian.

21. We need to determine whether there exist scalars s,t,u not all zero such that

$$s(\mathbf{i}+\mathbf{j}+\mathbf{k})+t(2\mathbf{i}-\mathbf{j})+u(3\mathbf{i}-\mathbf{j}-\mathbf{k})=\mathbf{0}$$

$$(s+2t+3u)\,\mathbf{i}+(s-t-u)\,\mathbf{j}+(s-u)\,\mathbf{k}=\mathbf{0}.$$

The only solution of the system

$$s+2t+3u=0, \quad s-t-u=0, \quad s-u=0$$

is $s=t=u=0$. Thus, the vectors are not coplanar.

22. coplanar since $1(\mathbf{j}-\mathbf{k})-1(3\mathbf{i}-\mathbf{j}+2\mathbf{k})+1(3\mathbf{i}-2\mathbf{j}+3\mathbf{k})=\mathbf{0}$

23. By (12.7.7), $d(P,p)=\dfrac{|2(2)+4(-1)-(3)+1|}{\sqrt{4+16+1}}=\dfrac{2}{\sqrt{21}}=\dfrac{2}{21}\sqrt{21}.$

24. $d=\dfrac{|8(3)-2(-5)+2-5|}{\sqrt{8^2+(-2)^2+1^2}}=\dfrac{31}{\sqrt{69}}$

28. $\overrightarrow{P_1P_2}=(1,-3,-2), \quad \overrightarrow{P_1P_3}=(-1,1,0), \quad \mathbf{N}=\overrightarrow{P_1P_2}\times\overrightarrow{P_1P_3}=(2,2,-2)$

$\implies 2(x-1)+2(y-1)-2(z-1)=0$ or $x+y-z-1=0$

29. $\overrightarrow{P_1P}=(x-3)\,\mathbf{i}+(y+4)\,\mathbf{j}+(z-1)\,\mathbf{k}, \quad \overrightarrow{P_1P_2}=6\mathbf{j}, \quad \overrightarrow{P_1P_3}=-4\mathbf{i}+5\mathbf{j}-3\mathbf{k}.$

Therefore

$$(\overrightarrow{P_1P_2} \times \overrightarrow{P_1P_3}) = 6\mathbf{j} \times (-4\mathbf{i} + 5\mathbf{j} - 3\mathbf{k}) = -18\mathbf{i} + 24\mathbf{k}$$

35. We set $x = 0$ and find that $P_0(0,0,0)$ lies on the line of intersection. As normals to the plane we use

$$\mathbf{N}_1 = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k} \quad \text{and} \quad \mathbf{N}_2 = -3\mathbf{i} + 4\mathbf{j} + \mathbf{k}.$$

Note that

$$\mathbf{N}_1 \times \mathbf{N}_2 = (\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}) \times (-3\mathbf{i} + 4\mathbf{j} + \mathbf{k}) = -10\mathbf{i} - 10\mathbf{j} + 10\mathbf{k}.$$

We take $-\frac{1}{10}(\mathbf{N}_1 \times \mathbf{N}_2) = \mathbf{i} + \mathbf{j} - \mathbf{k}$ as a direction vector for the line through $P_0(0,0,0)$. Then

$$x(t) = t, \quad y(t) = t, \quad z(t) = -t.$$

36. Using the hint, we find $P(0, \frac{1}{2}, -\frac{3}{2})$ on the line of intersection.

For the direction vector, consider $N_1 \times N_2 = (\mathbf{i} + \mathbf{j} + \mathbf{k}) \times (\mathbf{i} - \mathbf{j} + \mathbf{k}) = 2\mathbf{i} - 2\mathbf{k}$, so we can use $\mathbf{d} = \mathbf{i} - \mathbf{k}$. Thus

$$x(t) = t, \quad y(t) = \frac{1}{2}, \quad z(t) = -\frac{3}{2} - t.$$

37. Straightforward computations give us

$$l\colon x(t) = 1 - 3t, \quad y(t) = -1 + 4t, \quad z(t) = 2 - t$$

and

$$p\colon x + 4y - z = 6.$$

Substitution of the scalar parametric equations for l in the equation for p gives

$$(1 - 3t) + 4(-1 + 4t) - (2 - t) = 6 \quad \text{and thus} \quad t = 11/14.$$

Using $t = 11/14$, we get $x = -19/14, \quad y = 15/7, \quad z = 17/14.$

38. $l: x(t) = 4 - 2t, \quad y(t) = -3 + t, \quad z(t) = 1 + 2t \quad P: \; x + 4y - z = 6$

Note that $\mathbf{d} \cdot \mathbf{N} = (-2\mathbf{i} + \mathbf{j} + 2\mathbf{k}) \cdot (\mathbf{i} + 4\mathbf{j} - \mathbf{k}) = 0$, so the line is parallel to the plane, and since P_1 does not lie in the plane, l and P do not intersect.

46. (a) intercepts:

$(6,0,0), \; (0,3,0), \; (0,0,2)$

(b) traces:

in the x,y-plane: $x + 2y = 6$

in the x,z-plane: $x + 3z = 6$

in the y,z-plane: $2y + 3z = 6$

(c) unit normals: $\pm\dfrac{1}{\sqrt{14}}(\mathbf{i} + 2\mathbf{j} + 3\mathbf{k})$

47. (a) intercepts:

$(4,0,0), \; (0,5,0), \; (0,0,2)$

(b) traces:

in the x,y-plane: $5x + 4y = 20$

in the x,z-plane: $x + 2z = 4$

in the y,z-plane: $2y + 5z = 10$

(c) unit normals: $\pm\dfrac{1}{\sqrt{141}}(5\mathbf{i} + 4\mathbf{j} + 10\mathbf{k})$

51. $\dfrac{x}{2} + \dfrac{y}{5} + \dfrac{z}{4} = 1$