

## Chapter 13: VECTOR CALCULUS

**VECTOR FUNCTIONS:** Let  $f_1(t)$ ,  $f_2(t)$ ,  $f_3(t)$  be functions defined on some  $t$ -interval  $I$ . For each  $t \in I$ , form the vector

$$\mathbf{f}(t) = f_1(t)\mathbf{i} + f_2(t)\mathbf{j} + f_3(t)\mathbf{k}.$$

$\mathbf{f}$  is called a *vector-valued function* or, more simply, a *vector function*. Note: there is nothing special about 3 dimensions here; we can have vector functions in any number of coordinates. In particular, we will also see many examples in two dimensions.

**1. CALCULUS:** Let  $\mathbf{f}(t) = f_1(t)\mathbf{i} + f_2(t)\mathbf{j} + f_3(t)\mathbf{k}$ .

**A. Limits:** Let  $c \in I$ . Then

$$\begin{aligned} \lim_{t \rightarrow c} \mathbf{f}(t) = \mathbf{L} = L_1\mathbf{i} + L_2\mathbf{j} + L_3\mathbf{k} \quad & \text{if and only if} \quad \lim_{t \rightarrow c} \|\mathbf{f}(t) - \mathbf{M}\| = 0 \\ & \text{if and only if} \quad \lim_{t \rightarrow c} f_1(t) = L_1, \quad \lim_{t \rightarrow c} f_2(t) = L_2, \quad \lim_{t \rightarrow c} f_3(t) = L_3 \end{aligned}$$

The usual limit theorems hold: limit of a sum, difference, etc.

**B. Derivatives:**  $\mathbf{f}$  is *differentiable* at  $t$  if and only if

$$\lim_{h \rightarrow 0} \frac{1}{h} [f(t+h) - f(t)] \quad \text{exists.}$$

If the limit exists, it is called the *derivative of  $\mathbf{f}$  at  $t$*  and is denoted  $\mathbf{f}'(t)$ . Because of the properties of limits indicated in (A),

$$\mathbf{f}'(t) = f'_1(t)\mathbf{i} + f'_2(t)\mathbf{j} + f'_3(t)\mathbf{k}.$$

**C. Integrals:**

$$\int \mathbf{f}(t) dt = \int f_1(t) dt \mathbf{i} + \int f_2(t) dt \mathbf{j} + \int f_3(t) dt \mathbf{k}$$

**2. GEOMETRY:** Let  $x = x(t)$ ,  $y = y(t)$ , and  $z = z(t)$  be differentiable functions on some  $t$ -interval  $I$  and let  $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$ . The tip of the vector  $\mathbf{r}$  traces out a curve  $C$  in space. The equations:

$$x = x(t), \quad y = y(t), \quad z = z(t)$$

are *parametric equations* for  $C$ . Also,  $C$  is an *oriented curve*; the "positive direction" on  $I$  induces a positive direction on  $C$ .

**A. Tangent Vector:** The vector

$$\mathbf{r}'(t) = x'(t)\mathbf{i} + y'(t)\mathbf{j} + z'(t)\mathbf{k},$$

if not  $\mathbf{0}$ , is a direction vector for the line tangent to  $C$  at the point  $(x(t), y(t), z(t))$  on  $C$ ;  $\mathbf{r}'(t)$  points in the direction of increasing  $t$ .

**B. Unit Tangent Vector:** If  $\mathbf{r}'(t) \neq \mathbf{0}$ , then

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|}$$

is the *unit tangent vector*.

**C. Principal Normal Vector:** If  $\mathbf{T}'(t) \neq \mathbf{0}$ , then

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|}$$

is the *principal normal vector*;  $\mathbf{T} \perp \mathbf{N}$ .

**D. Osculating Plane:** The plane determined by  $\mathbf{T}$  and  $\mathbf{N}$  is called the *osculating plane*. In particular, choose  $c \in I$ . Then  $P(x(c), y(c), z(c))$  is a point on the curve  $C$  and  $\mathbf{T}(c) \times \mathbf{N}(c)$  is a normal vector for the osculating plane at  $P$ . This is the information needed to write an equation for the osculating plane.

**3. MECHANICS:** Let  $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$  denote the position at time  $t$  of an object. As  $t$  ranges over the interval  $I$ , the object moves along the curve  $C$ .

**A. Velocity Vector:** The vector

$$\mathbf{v}(t) = \mathbf{r}'(t) = x'(t)\mathbf{i} + y'(t)\mathbf{j} + z'(t)\mathbf{k}$$

is called the *velocity vector* and

$$\|\mathbf{r}'(t)\| = \sqrt{[x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2}$$

is the *speed* of the object at time  $t$ .

**B. Acceleration Vector:** The vector

$$\mathbf{a}(t) = \mathbf{v}'(t) = \mathbf{r}''(t) = x''(t)\mathbf{i} + y''(t)\mathbf{j} + z''(t)\mathbf{k}$$

is called the *acceleration vector*.

**4. ARC LENGTH/DISTANCE:** Let  $a, b \in I$ ,  $a < b$ . The length of the curve  $C$  for  $a \leq t \leq b$  is given by:

$$L(C) = \int_a^b \sqrt{[x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2} dt = \int_a^b \|\mathbf{r}'(t)\| dt$$

**Distance:** The distance  $s(t)$  traveled by a particle moving along  $C$  over the time interval  $[a, t]$  is given by:

$$s(t) = \int_a^t \sqrt{[x'(u)]^2 + [y'(u)]^2 + [z'(u)]^2} du$$

The derivative of distance with respect to time,  $ds/dt$ , is

$$\frac{ds}{dt} = \sqrt{[x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2} = \|\mathbf{r}'(t)\|,$$

see Section 5.2 

and this is the speed of the object as noted above.

**5. CURVATURE:** The *curvature* of a curve is a measure of the rate at which the curve is "curving." The curvature of  $C$  is the magnitude of the change of the unit tangent vector with respect to arc length:

$$\kappa = \left\| \frac{\mathbf{T}}{ds} \right\| = \frac{\|\mathbf{T}/dt\|}{ds/dt} = \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|}$$

**Special Cases:**

(1)  $C : y = f(x)$  the graph of a function:

$$\kappa = \frac{|y''(t)|}{(1 + [y']^2)^{3/2}}$$

(2)  $C : x = x(t), y = y(t)$  a plane curve defined parametrically"

$$\kappa = \frac{|x'(t)y''(t) - y'(t)x''(t)|}{([x']^2 + [y']^2)^{3/2}}$$

**6. TANGENTIAL AND NORMAL COMPONENTS OF ACCELERATION:** The acceleration vector

$$\mathbf{a}(t) = \mathbf{a}(t) = x''(t)\mathbf{i} + y''(t)\mathbf{j} + z''(t)\mathbf{k}$$

is a linear combination of the unit tangent and principal normal vectors; i.e., the acceleration vector lies in the osculating plane:

$$\mathbf{a}(t) = a_T \mathbf{T} + a_N \mathbf{N}$$

The coefficients  $a_T$  and  $a_N$  are called the *tangential* and *normal* components of acceleration;  $a_T$  and  $a_N$  are given by:

$$a_T = \frac{d^2s}{dt^2}, \quad a_N = \kappa \left( \frac{ds}{dt} \right)^2$$