Main topics (Math 7350)

Most items below come with a definition, examples and maybe proof, even if not mentioned explicitly. M^d denotes a manifold of dimension d, and $k \geq 1$.

1. Review of multivariable calculus (on open subsets of Euclidean spaces):

   – higher order derivatives, Taylor polynomials

   – the Inverse Function, Implicit Function and Rank theorems; "normal" forms for such maps (i.e., the simplest form, up to local diffeomorphisms)

2. Definition of manifolds (with and without boundary) We restrict ourselves to second countable Hausdorff spaces.

   C^k-charts. (Maximal) atlas, differential structure.

   Topology through charts (if the manifold is not already a topological space).

   New charts & manifolds from old.

   Theorem (Whitney): Each C^k manifold admits a compatible C^∞ structure. All these C^∞ structures are smoothly diffeomorphic. [Thus, could assume from now on that all manifolds are smooth.]

3. Differentiable maps between manifolds.

   Critical, singular etc. points of a map.

   Submersions, immersions, embeddings.

   The Inverse Function, Implicit Function and Rank theorems on manifolds.

4. Submanifolds.

   Preimage of a regular point is a submanifold.

   Image of an immersion or embedding.

5. The tangent space: in local coordinates (and associated basis), through curves, or as derivations (for smooth manifolds).

   The differential of a map (in each of the above representations).

6. General vector bundles: local trivializations, the transition functions, the compatibility condition.

   The tangent bundle.

   Vector fields (in various descriptions).

7. ODE's, in $\mathbb{R}^n$ and on a manifold.

   The flow (local/global one-parameter group) associated to a vector field.

8. The Lie derivative (of a tensor field) with respect to a vector field. [The Lie derivative is actually associated to the one-parameter group determined by the vector field.]

   The bracket of two vector fields, the Lie algebra of smooth vector fields.

9. Distributions, integrability.

   If $X_1, X_2, \ldots, X_d$ are commuting vector fields which are linearly independent at each point, then (locally) there is a chart such that $X_k = \frac{\partial}{\partial x_k}$ for $1 \leq k \leq d$.

   The Frobenius theorem (the proof in Spivak is more conceptual).

10. Tensors, alternating forms for a finite dimensional vector space.

    Bases for tensors and forms. The wedge product.

11. The bundle of (differential) forms of a manifold.
    The exterior differential of a form; $d^2 = 0$.
    Pull-back of a form, its behavior with respect to the wedge product and exterior differential.
12. Closed and exact forms. The Poincaré Lemma.
13. Partitions of unity (including locally finite covers, etc.).
14. Orientation of a vector space.
    Orientation of a manifold. M^n is orientable iff there exists a nowhere vanishing n-form.
    Orientation induced on the boundary of an oriented manifold.
15. Integration of k-forms on singular k-chains.
    The boundary of a chain; $\partial^2 = 0$.
    The Stokes theorem for chains.
16. Integration of compactly supported n-forms on an oriented n-dimensional manifold.
    The Stokes theorem for manifolds.
17. Embedding of compact manifolds in $\mathbb{R}^N$.
    The "medium" Whitney embedding theorem (1936): $M^n \hookrightarrow \mathbb{R}^{2n+1}$.