Homework 1

Updated January 26, 2009; footnotes added Feb. 3, 2009

Due date: February 5, 2009 (Thursday)

DO NOT HAND IN:
Please think about these problems, but do not hand in the solutions.

Exercises 1.1, 1.2 and 1.3.

TO HAND IN:
Please write solutions to these problems in sufficient detail.

1. Prove that the map we used to define charts of the real projective space (see Example 1.3),

$$\widetilde{\varphi}_1 : \widetilde{U}_1 \subset \mathbb{R}^{n+1} \to \mathbb{R}^n, \qquad \widetilde{\varphi}_1(x^1, x^2, \ldots, x^{n+1}) = \left(\frac{x^2}{x^1}, \frac{x^3}{x^1}, \ldots, \frac{x^{n+1}}{x^1}\right)$$

   is a quotient map,[1] where $\widetilde{U}_1 = \{(x^1, x^2, \ldots, x^{n+1}) \in \mathbb{R}^{n+1} \mid x^1 \neq 0\}$.

   You can work with $n = 2$ if you prefer.

2. Explain how second countability of a topological manifold M implies that it can be covered by countably many coordinate domains (we used this to prove Lemma 1.6). [2]

3. This is Problem 1-5 from Lee, stated for S^2. See that problem for more explanations and a drawing. [3]

   The *stereographic projections* on the two-sphere S^2 are

$$\varphi_+ : S^2 \setminus \{(0,0,1)\} \to \mathbb{R}^2, \quad \varphi_+(x,y,z) = \left(\frac{x}{1-z}, \frac{y}{1-z}\right)$$

   and

$$\varphi_- : S^2 \setminus \{(0,0,-1)\} \to \mathbb{R}^2, \quad \varphi_-(x,y,z) = \left(\frac{x}{1+z}, \frac{y}{1+z}\right)$$

   (φ_+ from the North pole, φ_- from the South pole).

   (a) Explain why these maps are homeomorphisms. [By "symmetry", it suffices to treat only one of them.]

[1] The topology on both the domain and range of $\widetilde{\varphi}_1$ are known here (the standard ones), so one has to check that $\widetilde{\varphi}_1$ is a quotient map *w.r.t. these topologies.*

[2] You can invoke a general result, but I would prefer that you give the full argument – it is not too complicated.

[3] "Explain" in (a) is meant as "prove". To show that a map is smooth it suffices to rely on arguments that we used in class (e.g., composition of smooth maps is smooth); usually there is no need to compute derivatives and show that they are continuous etc.

(b) Check that these two charts are smoothly compatible.

(c) Check that these charts are smoothly compatible with the charts introduced in Example 1.2 of Lee. (Enough to check one of the stereographic projection charts against one chart from the other set.)