Homework 3

Updated Feb. 23, 2009

Due date: Tuesday, March 3, 2009

HAND IN: Please write your solutions in sufficient detail.

1. Since a linear invertible transformation $A \in \mathrm{GL}(\mathbb{R}^3)$ takes lines of $\mathbb{R}^3$ to lines of $\mathbb{R}^3$, it induces a map $f = \widehat{A} : \mathbb{RP}^2 \to \mathbb{RP}^2$. [More generally, we saw in Homework 2 that homogeneous smooth maps induce smooth maps of the projective spaces if they never take a nonzero point to zero.] Consider the mapping of this type given by
$$f([x : y : z]) = [y : y - z : x - z].$$
Compute the push-forward (a.k.a. differential) of this mapping at the point $p = [1 : 0 : 0]$. Use the atlas with three charts described in the course (see Example 1.3).

   More precisely:

   - pick the (unique) charts that are defined around p and $q := f(p)$;
   - these charts give canonical bases of $T_p\mathbb{RP}^2$ and $T_q\mathbb{RP}^2$;
   - compute the linear transformation $df = f_* : T_p\mathbb{RP}^2 \to T_q\mathbb{RP}^2$ with respect to these bases.

2. Problem 3-1 from the textbook.

DO NOT HAND IN:

3. Problems 3-2 and 3-3 from the textbook.

4. Problem 3-4 from the textbook (using curves to describe tangent vectors).

5. Since the inclusion of the unit sphere S^2 into $\mathbb{R}^3$, $i : S^2 \to \mathbb{R}^3$, is a smooth map, it induces a linear map $i_* : T_pS^2 \to T_p\mathbb{R}^3$ which is actually one-to-one. Thus, one can identify in a natural way T_pS^2 with a subspace of $T_p\mathbb{R}^3$, as the geometric intuition suggests. This inclusion is easier to see if we define tangent vectors via curves, instead of derivations. [That $T_pN \subset T_pM$ is true for any submanifold $N \subset M$, to be seen later.]

   Consider the stereographic chart
$$\varphi = \varphi_+ : S^2 \setminus \{(0,0,1)\} \to \mathbb{R}^2, \quad (x^1, x^2) = \varphi(x,y,z) = \left(\frac{x}{1-z}, \frac{y}{1-z}\right)$$
and the point $p = (a,b,c) \in S^2$, $c \neq 1$. Compute the vectors corresponding in $T_p\mathbb{R}^3$ to the basis of T_pS^2 determined by φ,
$$\frac{\partial}{\partial x^1}|_p \text{ and } \frac{\partial}{\partial x^2}|_p.$$