

# Geometry of Manifolds – a preview\*

These notes are an informal presentation of a few Differential Geometry topics.

## Why manifolds?

The “simplest” higher dimensional objects are the vector spaces.

*manifolds* versus *vector spaces*  $\iff$  *curves* versus *straight lines*

The vector spaces are the simplest manifolds. Many objects that one encounters in modern mathematics are manifolds.

Here are two examples.

The unit sphere in  $\mathbb{R}^3$  is a manifold; however, locally it looks very much like a plane (no wonder people thought for a while that the Earth is flat). ***This is, roughly speaking, the general idea of a manifold: it can be “modeled” locally by a vector space.***

According to Einstein’s General Theory of Relativity, our physical world (i.e., the “space-time”) is a four dimensional manifold, which is determined (shaped) by matter. Objects (e.g., planets and stars) move on the “shortest” trajectories, called *geodesics*. As soon as matter is present, the universe differs from the canonical  $\mathbb{R}^4$ , and the geodesics are not always straight lines. In our solar system, the planets move on geodesics (of space-time) whose space-component can be well approximated by the ellipses etc. predicted by Newton’s theory of gravity (Newton developed his theory to work in the flat  $\mathbb{R}^3$ ).

In some cases it is natural to endow manifolds with additional structures. E.g., the natural set-up for classical mechanics are *symplectic manifolds*. Surfaces in  $\mathbb{R}^3$  inherit a *Riemannian structure*, which describes the distance. On the other hand, relativity is described using the Minkowski metric.

## 1 Manifolds

Unless specified otherwise, the manifolds are without boundary.

The following is a correct statement, although not really a definition:

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**“Definition” 1.1** An  $n$ -dimensional manifold is a Hausdorff topological space that is **locally diffeomorphic** to (an open subset of)  $\mathbb{R}^n$ .

For convenience we will consider manifolds that are second countable (and connected).

**Remark 1.2** We describe/recall briefly some of the notions mentioned above.

- A map  $\varphi : U \rightarrow V$  is a  $C^r$ -**diffeomorphism** if it is  $C^r$ , it is invertible, and its inverse is also  $C^r$ . [We leave it vague what  $U$  and  $V$  are; the familiar case is when they are open sets in the vector spaces  $\mathbb{R}^k$ .]

The **order of smoothness**  $r$  can be from 1 to  $\infty$ . [For  $r = 0$ , the corresponding maps are called homeomorphisms, and we speak of *topological manifolds*.]

- If  $U \subset \mathbb{R}^n$  and  $V \subset \mathbb{R}^m$ , then a map  $F : U \rightarrow V$  is given by  $m$  functions  $f_i : U \rightarrow \mathbb{R}$ ,  $F = (f_1, f_2, \dots, f_m)$ , each function having  $n$  variables.
- A mapping  $F = (f_1, f_2, \dots, f_m)$  is  $C^r$  if each of its components is  $C^r$ .
- If  $U \subset \mathbb{R}^n$  is an open set, then a function  $f : U \rightarrow \mathbb{R}$  is  $C^r$  if all of its partial derivatives up to order  $r$  exist, and these derivatives are continuous.

**Definition 1.3** A property is **local** if it is determined by information on a neighborhood of each point. E.g.: the derivative of a function, or the radius of curvature of a curve, are local quantities.

A property is **pointwise** if it is determined by the value only at the point (e.g., the value of a function is a pointwise quantity).

A property is **global** if it is determined by the values on the whole set (e.g., the value of the integral  $\int_U f dx$ ).

Now we can expand what “Definition” 1.1 says: a manifold is a topological space  $M$  so that each point  $p \in M$  has a neighborhood  $U$  which is diffeomorphic to an open subset of  $\mathbb{R}^n$ . [Attention: we still did not define all the terms.]

In other words: **as far as differentiability is concerned**, locally an  $n$ -dimensional manifold is **the same** as an open subset of  $\mathbb{R}^n$ .

## 1.1 Examples of manifolds: curves and surfaces

- (a) One-dimensional manifolds  $\equiv$  curves

Up to diffeomorphism, there are only two *compact* connected 1-dimensional manifolds:

- (1) the circle, a manifold without boundary:

$$S^1 := \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$$

(the standard choice is the unit circle).

- (2) the closed interval, a manifold with boundary:

$$I = [0, 1]$$

(we take this interval only for convenience).

(b) Two-dimensional manifolds  $\equiv$  surfaces

(1) The unit sphere:

$$S^2 := \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$$

(2) All **orientable** compact surfaces are spheres with handles attached:

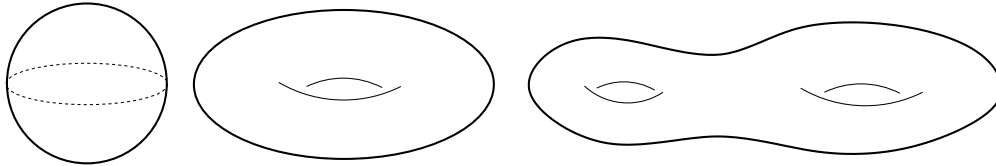


Figure 1: Sphere with zero, one, and two handles.

(3) The Möbius strip is a **non-orientable** compact surface with boundary, which can be described either as

– glue the two ends of a strip of paper, after turning one end by  $180^\circ$

or

– the unit square  $I \times I$  subject to the identifications

$$(0, y) \equiv (1, 1 - y) \quad (\text{glue points on left and right edges “across”})$$

As described above, one can easily realize this surface in  $\mathbb{R}^3$ .

(4) The Klein bottle is a **non-orientable** compact surface without boundary, which can be described either as

– two Möbius strips with the boundaries glued together

or

– the unit square  $I \times I$  subject to the identifications:

$$(x, 0) \equiv (x, 1) \quad (\text{glue top and bottom edges, to get a cylinder})$$

$$(0, y) \equiv (1, 1 - y) \quad (\text{glue points on left and right edges “across”})$$

This surface cannot be *embedded* (i.e., realized) in  $\mathbb{R}^3$ .

## 1.2 General manifolds

One can define a manifold structure on an abstract set. For example:

- The set of all lines through the origin in  $\mathbb{R}^{n+1}$  is a manifold<sup>1</sup> that has dimension  $n$ . It is called the *real projective space* and denoted  $\mathbb{R}P^n$ .
- The set of all planes through the origin in  $\mathbb{R}^n$ , the “2-Grassmanian”  $G_2(\mathbb{R}^n)$ , is a manifold<sup>2</sup> of dimension  $2(n - 2)$ .

<sup>1</sup>Here is the reason why: given a line  $\ell$ , the lines nearby can be labeled by their intersection with a hyperplane perpendicular to  $\ell$  that does not contain the origin.

<sup>2</sup>Here is a hint why: given a two-plane  $E_0$ , pick a subspace  $F_0$  such that  $E_0 \oplus F_0 = \mathbb{R}^n$ ; then each two-plane  $E$  that intersects trivially  $F_0$  can be identified with the graph of a linear map from  $E_0$  into  $F_0$  (the plane  $E_0$  corresponds to the zero map). Thus, a neighborhood of  $E_0$  in  $G_2(\mathbb{R}^n)$  can be “labeled” by the linear maps from a 2-dimensional space to a  $(n - 2)$ -dimensional space, hence  $(n - 2) \times 2$  matrices.

- $O(n) := \{A \in \text{Mat}_{n \times n}(\mathbb{R}) \mid AA^t = I\}$ , the set of all orthogonal linear transformations of  $\mathbb{R}^n$ , is a manifold and actually a **Lie group**.

However, one can also realize the “abstract manifolds” as **sub-manifolds** of the vector space  $\mathbb{R}^N$ . Below are some results, not in their strongest form.

The statement *M can be embedded into N* means that one can realize the manifold *M* as a sub-manifold of *N*.

**Theorem 1.4 (Whitney, easiest version)** *Any compact manifold can be embedded into  $\mathbb{R}^N$  for some large enough  $N$ .*

**Theorem 1.5 (Whitney 1936, medium version)** *Any compact  $n$ -dimensional manifold can be embedded into  $\mathbb{R}^{2n+1}$ .*

**Theorem 1.6 (Whitney 1944, difficult version)** *Any compact  $n$ -dimensional manifold can be embedded into  $\mathbb{R}^{2n}$ .*

Thus, the Klein bottle can be embedded into  $\mathbb{R}^4$ .

## 2 Vector bundles

Each manifold *M* has a naturally defined **tangent bundle** *TM*. The vectors tangent to *M* at a point  $p \in M$  form a vector space,  $T_pM$ .

For example, if we consider the unit sphere  $S^2 \subset \mathbb{R}^3$ , then one can identify the set of tangent vectors at a point  $p$ ,  $T_pS^2$ , with the plane tangent to the sphere at that point. [To be precise, in order to make this plane a vector space, it should be translated to the origin.]

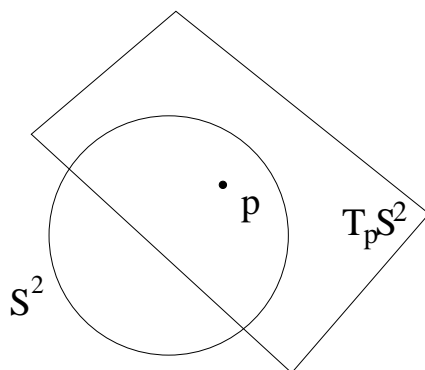


Figure 2: The tangent bundle to the 2-sphere.

Similarly, one can consider other bundles over a manifold. For example, one can consider the line bundle *L* over  $S^2 \subset \mathbb{R}^3$  given at  $p \in S^2$  by the line normal to the sphere at  $p$ .

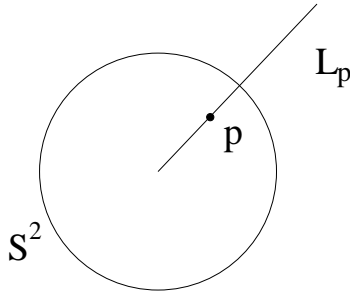


Figure 3: A line bundle over the 2-sphere.

### 3 Integration on manifolds

Without any additional data, one can integrate **naturally** on an *orientable* manifold of dimension  $n$  only  $n$ -forms. In *local coordinates*<sup>3</sup>, an  $n$ -form looks like

$$f(x_1, x_2, \dots, x_n) dx_1 \wedge dx_2 \wedge \dots \wedge dx_n$$

Here “naturally” means that the result of the integration does not depend on the choice of local coordinates.

One central result, which generalizes Green’s Theorem discussed in Calculus III and is related to many conservation laws in Physics, is:

**Theorem 3.1 (Stokes’ Theorem)** *Let  $M$  be an orientable manifold (possibly with boundary) of dimension  $k$  and  $\omega$  a compactly supported  $(k - 1)$ -form. Then*

$$\int_M d\omega = \int_{\partial M} \omega.$$

Here  $d\omega$  denotes the *exterior derivative* of the form  $\omega$ .

### 4 Riemannian geometry

One way to introduce distance on a manifold  $M$  is to endow it with a **Riemannian metric**, which is an inner product on each tangent space  $T_p M$ . Using this one can associate length to each curve, much in the same way as we do it for curves in  $\mathbb{R}^3$ . It also determines a *volume form*, and thus *allows to integrate functions* on  $M$ . Moreover, the Riemannian structure is inherited naturally by the submanifolds of  $M$ .

Given a Riemannian structure, one can speak of curves that minimize distance locally, called **geodesics**. One can also introduce **curvature**.

### 5 Global results

The quantities discussed so far are of local character (all are defined considering neighborhoods). Therefore, they do not “see” the whole manifold. But certain quantities are constrained by global data. We give three examples; the last two results exhibit the connection between local and global quantities.

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<sup>3</sup>Explain what these are.

Given a manifold  $M$  one can use forms to define its *de Rham cohomology*  $H^*(M, \mathbb{R})$ , which is a *topological invariant*.<sup>4</sup>

The recently proved *Poincaré conjecture* asked whether certain *topological* information identifies completely the three-sphere  $S^3$  within the family of three-manifolds<sup>5</sup>. The positive answer was obtained by Grigori Perelman following the program of Richard Hamilton, about 100 years after Poincaré’s question.

**Theorem 5.1 (Gauss-Bonnet)** *Given a compact two-dimensional Riemannian manifold  $M$  without boundary,*

$$\int_M K dA = 2\pi\chi(M)$$

where  $K$  is the Gaussian curvature,  $dA$  is the area element, and  $\chi(M)$  is the Euler characteristic,

$$\chi(M) := \sum_{k=0}^{\dim M} (-1)^k \dim H^k(M, \mathbb{R}).$$

The remarkable fact about this formula is that the curvature  $K$  on the left-hand side (determined by the Riemannian structure) can be changed by “bending” the surface, however *its total integral remains constant*, and is a priori determined by the topology of the surface.

For example, the sphere of radius  $R$  in  $\mathbb{R}^3$  has constant curvature  $K \equiv 1/R^2$ , area element given by the usual surface area, and Euler characteristic  $\chi(S^2) = 2$ ; indeed

$$\frac{1}{R^2} \times (4\pi R^2) = 2\pi \times 2.$$

No matter how we deform the sphere without “tearing it”, the right-hand side remains  $4\pi$ ; however, the curvature will change once the sphere is deformed.

**Theorem 5.2 (Poincaré-Hopf index theorem)** *Let  $M$  be a compact manifold  $M$  without boundary. If  $v$  is a smooth vector field that has only isolated zeroes (points where the vector field vanishes) then*

$$\sum_{\{p \in M | v(p)=0\}} \text{index}_v(p) = \chi(M)$$

The left hand side is of local nature (we skip the definition of the index for the moment), whereas the right-hand side is a global quantity.

For example, since the Euler characteristic of the two-sphere is non-zero, the above theorem shows that any smooth vector field on  $S^2$  must vanish somewhere (which is another way to state the so-called “Hairy Ball Theorem”: one cannot comb hair on a two-sphere without creating a vortex or other discontinuity).

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<sup>4</sup>That is, if two manifolds are *homeomorphic*, then their cohomologies coincide. Say a few words about algebraic topology?

<sup>5</sup>Is every simply connected compact 3-manifold (without boundary) homeomorphic to a 3-sphere?