Manifolds: Topological versus Differentiable*

Here are some facts about the differential (smooth) structures that a topological manifold of dimension n can support, and related results.

Unless stated otherwise, all manifolds are without boundary. Counting (e.g., uniqueness) is meant in the appropriate sense (up to smooth diffeomorphisms for smooth manifolds).

See also the remarks on pages 14 and 37 of the textbook.

(a) $n \leq 3$

- Each n-dimensional topological manifold has a unique smooth structure (J. Munkres, E. Moise).

(b) $n \geq 4$

- For each n there is a connected compact topological manifold that does not admit a smooth structure.
- Each n-dimensional compact connected topological manifold admits at most countably many different smooth structures.
- S^7 admits exactly 28 different smooth structures (J.W. Milnor and M.A. Kervaire).
- Classification of smooth (even compact) manifolds of dimension $n \geq 4$ is very hard.
- Does S^4 or $\mathbb{CP}^2$ admit more than one smooth structures?

(c) The Poincaré conjecture (1904): a compact simply-connected[1] smooth manifold is homeomorphic to the sphere of that dimension.

- $n \geq 5$ proved by S. Smale (1961).
- $n = 4$ proved by M.H. Freedman (1982).
- $n = 3$ proved by G. Perelman (2002–2003), using the Ricci flow method of R.S. Hamilton.

  This makes the "geometrization conjecture" of W. Thurston close to being proved[2]: there are eight standard Riemannian models in dimension 3 (compare to the dimension-2 case below, where there are three models).

*Last updated: Jan 27, 2009; only limited claim of accuracy is made.

[1] A (path connected) topological space X is *simply-connected* if its fundamental group, $\pi_1(X)$, is trivial: any closed path in X can be continuously "shrunk" to a point. The 1-sphere is not simply-connected but all higher dimensional spheres are.

[2] Might be already proven, using Perelman's work.

(d) $n = 1$

- Any 1-dimensional connected smooth manifold is diffeomorphic to either $\mathbb{R}$ or S^1 with the canonical structure.

(e) $n = 2$

- Any 2-dimensional connected compact oriented smooth manifold is diffeomorphic to the sphere S^2 with zero or more handles attached.
- Any 2-dimensional compact manifold M admits a Riemannian metric with makes it locally isometric to one of the following (because they are quotients of the manifolds listed below, under a free discrete group action):
  (1) S^2 with the canonical metric (i.e., the one inherited from $S^2 \subset \mathbb{R}^3$); this has constant $+1$ curvature.
  (2) $\mathbb{R}^2$ with the Euclidean metric; this has constant 0 curvature.
  (3) The hyperbolic plane, $\mathbb{H}^2$, with the Poincaré metric; this has constant -1 curvature.

  The model above that applies to M is determined by the sign of its Euler characteristic,
  $$\chi(M) := \sum_{i \geq 0} (-1)^i \dim H^i(M, \mathbb{R}).$$
  [Recall that by Gauss-Bonnet $\int_M K dA = 2\pi\chi(M)$, so the Euler characteristic determines the Gauss curvature if the latter is constant.]
- For example, the only compact 2-dimensional smooth manifolds that fall in the first case above are S^2 and $\mathbb{RP}^2$; in the second case above are the 2-torus, $S^1 \times S^1$, and the Klein bottle.

  The surfaces of genus at least 2 (in the orientable case, these are the sphere with at least two handles) admit a metric with constant curvature -1.
- The only smooth non-compact manifolds that admit a complete metric[3] of constant zero curvature are $\mathbb{R}^2$, $\mathbb{R} \times S^1$ and the Möbius strip (without boundary); there are no such manifolds for curvature $+1$. The hyperbolic plane is such an example for curvature -1.

(f) $\mathbb{R}^n$

- For $n \neq 4$, $\mathbb{R}^n$ admits a unique smooth structure.
- $\mathbb{R}^4$ admits uncountably many non-diffeomorphic smooth structures (S.K. Donaldson and M.H. Freedman, 1984)

[3] A Riemannian metric is complete if each geodesic can be extended to infinite time. For example, $\mathbb{R}^n$ and any compact manifold are complete, but $\mathbb{R}^n \setminus \{0\}$ is not complete with the canonical metric because some geodesics (straight lines in this case) run in the missing origin in finite time.