OPTIMAL FRAMES FOR ERASURES

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ABSTRACT. We study frames from the viewpoint of coding theory. We introduce a numerical measure of how well a frame reconstructs vectors when some of the frame coefficients of a vector are lost and then attempt to find and classify the frames that are optimal in this setting.

1. Introduction

In [8], [16] and [17] the family of uniform tight frames are studied from a coding theory viewpoint and these frames are shown to be optimal in some sense for one erasure. They then develop further properties of these frames including their robustness to more than one erasure.

In this paper we introduce a measure of how well a frame behaves under erasures and then seek optimal frames in this context. In some cases we are able to prove that, up to a natural equivalence, there exists a unique optimal frame and we are able to construct it. We introduce a family of frames that, when they exist, we prove are optimal. After writing a preliminary draft of this paper we learned that this family of frames was also being studied independently by Thomas Strohmer and Robert Heath[25] and we have incorporated a number of their observations into this paper.

We begin by recalling the basic definitions and concepts.

Let $\mathcal{H}$ be a Hilbert space, real or complex, and let $F = \{f_i\}_{i \in \mathbb{I}} \subset \mathcal{H}$ be a subset. We call F a *frame* for $\mathcal{H}$ provided that there are two constants

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$C, D > 0$ such that the inequality

$$C \cdot \|x\|^2 \leq \sum_{j \in \mathbb{I}} |\langle x, f_j \rangle|^2 \leq D \cdot \|x\|^2$$

holds for every $x \in \mathcal{H}$. When $C = D = 1$, then we call F a *normalized, tight frame.* Such frames are also called *Parseval frames* and this latter term is becoming more standard.

A frame F is called *uniform* or *equal-norm* provided there is a constant c so that $\|f\| = c$ for all $f \in F$ and we call F a *UNT frame* provided that it is uniform, normalized and tight.

The map $V : \mathcal{H} \to \ell_2(\mathbb{I})$ defined by

$$(Vx)_i = \langle x, f_i \rangle$$

is called the *analysis operator.* When F is a normalized, tight frame, then V is an isometry and the adjoint, V^* acts as a left inverse to V.

For the purposes of this paper we will only be concerned with finite dimensional Hilbert spaces and frames for these spaces that consist of finitely many vectors. When the dimension of $\mathcal{H}$ is k, then we will identify $\mathcal{H}$ with $\mathbb{R}^k$ or $\mathbb{C}^k$ depending on whether we are dealing with the real or complex case and for notational purposes regard vectors as columns. When we wish to refer to either case, then we will denote the ground field by $\mathbb{F}$. We shall let $\mathcal{F}(n, k)$ denote the collection of all normalized, tight frames for $\mathbb{F}^k$ consisting of n vectors and refer to such a frame as either a real or complex *(n,k)-frame,* depending on whether or not the field is the real numbers or the complex numbers. Thus, a uniform (n, k)-frame is a UNT frame for $\mathbb{F}^k$ with n vectors.

We shall often identify a frame with its analysis operator so that every (n, k)-frame is identified with the $n \times k$ isometry matrix V where the columns of V^* are the frame vectors.

Using some basic operator theory, it follows that $VV^* = (\langle f_j, f_i \rangle)$ is the Grammian(or correlation) matrix of the vectors and consequently, F is an

(n, k)-frame if and only if this matrix is a self-adjoint $n \times n$ projection of rank k.

Recall also that the rank of a projection is equal to its trace. Thus, when F is a uniform (n, k)-frame each of the diagonal entries of VV^* must be equal to k/n and so each frame vector must be of length $\sqrt{k/n}$.

Conversely, given an $n \times n$ self-adjoint projection P of rank k, we can always factor it as $P = VV^*$ for some $n \times k$ matrix V. It readily follows that $V^*V = I_k$ and hence V is the matrix of an isometry and so corresponds to an (n, k)-frame. Moreover, if $P = WW^*$ is another factorization of P, then there exists a unitary U such that $W^* = UV^*$ and hence the two corresponding frames differ by multiplication by this unitary. Thus, P determines a unique unitary equivalence class of frames. A projection P corresponds to a uniform (n, k)-frame if and only if all of its diagonal entries are k/n.

Finally, we wish to identify certain frames as being equivalent. Given frames $F = \{f_1, \dots, f_n\}$ and $G = \{g_1, \dots, g_n\}$, we say that they are *type I equivalent* if there exists a unitary(orthogonal matrix, in the real case) U such that $g_i = Uf_i$ for all i. If V and W are the analysis operators for F and G, respectively, then it is clear that F and G are type I equivalent if and only if $V = WU$ or equivalently, if and only if $VV^* = WW^*$. Thus, there is a one-to-one correspondence between $n \times n$ rank k projections and type I equivalence classes of (n, k)-frames.

We say that two frames are *type II equivalent* if they are simply a permutation of the same vectors and *type III equivalent* if they differ by multiplication by ± 1 in the real case and multiplication by complex numbers of modulus one, in the complex case.

Finally, we say that two frames are *equivalent* if they belong to the same equivalence class in the equivalence relation generated by these three equivalence relations. It is not hard to see that if F and G are frames with analysis operators V and W, respectively, then they are equivalent if and only if $UVV^*U^* = WW^*$ for some $n \times n$ unitary U that is the product of a permutation and a diagonal unitary(orthogonal matrix, in the real case).

We caution the reader that the equivalence relation that we have just defined is different than the equivalence relation that is often used. Often frames $\{f_i\}$ and $\{g_i\}$ are called equivalent provided that there is an invertible operator T such that $Tf_i = g_i$ for all i.

In [16] uniform (n, k)-frames are shown to exist by concretely exhibiting a particular UNT frame for each pair of integers (n, k). In the complex case, these are constructed using an n-th root of unity. In the real case, the formula involves sines and cosines and depends on whether n is even or odd. They call these frames the *harmonic tight frames.*

The following alternate proof of the existence of uniform (n, k)-frames, gives an algorithm to construct a uniform (n, k)-frame, starting with any (n, k)-frame.

Remark 1.1. *An algorithm for producing uniform frames.*

Let $F = \{f_1, \dots, f_n\}$ be a normalized tight frame, i.e., a (n, k)-frame, and let V be its analysis operator. Note that

$$\|f_1\|^2 + \dots \|f_n\|^2 = tr(VV^*) = k$$

since VV^* is a rank k projection. If F is already uniform we are done, otherwise there exists i and j such that $\|f_i\|^2 > k/n > \|f_j\|^2$.

If for any θ, we replace the vectors f_i, f_j by the vectors $g_i = cos(\theta)f_i - sin(\theta)f_j$ and $g_j = sin(\theta)f_i + cos(\theta)f_j$ and define $g_k = f_k$ for all other k, then $G = \{g_1, \dots g_n\}$ will also be an (n, k)-frame because its analysis operator W satisfies $W = UV$, for some unitary U and so W is also an isometry.

By choosing θ appropriately, we can insure that $\|g_i\|^2 = k/n$. Repeating this process at most $n - 1$ times we obtain a uniform (n, k)-frame.

This algorithm is essentially adopted from [22].

There is another place in the literature where uniform (n, k)-frames arise, but in a different guise. A finite subset of vectors $\{x_1, \dots, x_n\}$ on the unit sphere S^{k-1} in $\mathbb{R}^k$ is called a *spherical t-design*[13] provided that

$$\int_{S^{k-1}} p(x) d\omega(x) = 1/n \sum_{i=1}^{n} p(x_i)$$

for all polynomials of total degree at most t in the k coordinate variables, where $d\omega$ denotes unit normalized Lebesgue measure on the sphere.

Proposition 1.2. *A finite subset of vectors* $\{x_1, \dots, x_n\}$ *on the unit sphere* S^{k-1} *in* $\mathbb{R}^k$ *is a spherical 2-design if and only if* $\{\sqrt{k/n}x_1, \dots, \sqrt{k/n}x_n\}$ *is a uniform* (n, k)*-frame and* $\sum_{i=1}^n x_i = 0$.

Proof. Assume that we are given a spherical 2-design. Fix a vector y in $\mathbb{R}^k$ and let p be the degree 1 polynomial $p(x) = (< x, y >)$. Note that by the invariance of Lebesque measure on the sphere under orthogonal rotation the integral of p^2 over the unit sphere is $c\|y\|^2$ where c is a constant independent of y. Hence for every vector y we have that

$$c\|y\|^2 = 1/n \sum_{i=1}^{n} (< x_i, y >)^2$$

from which it follows that $\{\sqrt{c/n}x_1, \dots \sqrt{c/n}x_n\}$ is a uniform (n, k)-frame and hence $c = k$.

On the other hand, the integral of p over the unit sphere is seen to be equal to 0, for every y and hence $\sum_{i=1}^n < x_i, y >= 0$ from which it follows that $\sum_{i=1}^n x_i = 0$.

Conversely, assume that we are given that $\{\sqrt{k/n}x_1, \dots, \sqrt{k/n}x_n\}$ is a uniform (n, k)-frame whose vectors sum to 0. Since every degree 1 polynomial p is of the form $p(x) = p(0)+ < x, y >$ for some vector y we see that the sum and the integral are both $p(0)$ for every degree 1 polynomial and hence agree for first degree polynomials. For the function $q(x) =< x, y >^2$, both the sum and the integral are equal to the same multiple of the square of $\|y\|^2$. Thus, we see that they are equal for every degree 2 polynomial that is the square of a degree 1 polynomial. But every degree 2 polynomial is a linear combination of these special degree 2 polynomials and hence we have that the sum and integral agree for all degree 2 polynomials. □

2. Frames and Erasures

The idea behind treating frames as codes, is that given an original vector x in $\mathbb{F}^k$, and an (n, k)-frame with analysis operator V, one regards the vector Vx as an encoded version of x, which might then be somehow transmitted to a receiver and then decoded by applying V^*. Among all possible left inverses of V, we have that V^* is the unique left inverse that minimizes both the operator norm and Hilbert-Schmidt norm.

Suppose that in the process of transmission some number, say m, of the components of the vector Vx are lost, garbled or just delayed for such a long time that one chooses to reconstruct x with what has been received. In this case we can represent the received vector as EVx, where E is a diagonal matrix of m 0's and $n - m$ 1's corresponding to the entries of Vx that are, respectively, lost and received. The 0's in E can be thought of as the coordinates of Vx that have been "erased" in the language of [16].

There are now two methods by which one could attempt to reconstruct x. Either one is forced to compute a left inverse for EV or one can continue to use the left inverse V^* for V and accept that x has only been approximately reconstructed.

If EV has a left inverse, then the left inverse of minimum norm is given by $P^{-1}W^*$ where $EV = WP$ is the polar decomposition and $P = |EV| = (V^*EV)^{1/2}$. Thus, the minimum norm of a left inverse is given by p_{min}^{-1} where p_{min} denotes the least eigenvalue of P.

In the second alternative, the error in reconstructing x is given by

$$x - V^*EVx = V^*(I - E)Vx = (I - P^2)x = V^*DVx$$

where D is a diagonal matrix of m 1's and $n-m$ 0's. Thus, the norm of the error operator is $1 - p_{min}^2$.

Hence we see that, when a left inverse exists, the problems of minimizing the norm of a left inverse over all frames and of minimizing the norm of the error operator over all frames are really equivalent and are both acheived by maximizing the minimal eigenvalue of P.

In this section we pursue this second alternative, since this avoids the worry about whether or not a left inverse actually exists, and study the problem of finding a "best" frame for these circumstances. That is, a frame for which the norms of these error operators are in some sense minimized, independent of which erasures occur. Of course there are many ways that one could define "best" in this setting and we are only pursuing one reasonable possibility.

We shall continue to identify (n, k)-frames with $n \times k$ matrices, so that $\mathcal{F}(n, k)$ is identified with the set of $n \times k$ isometries and we wish to inductively define subsets of the (n, k)-frames.

To define these subsets, we first let $\mathcal{D}_m, 1 \leq m \leq n$ denote the set of $n \times n$ diagonal matrices with m 1's and $n - m$ 0's and for any isometry V in $\mathcal{F}(n, k)$ we set

$$d_m(V) = max\{\|V^* DV\| : D \in \mathcal{D}_m\},$$

where by the norm of a matrix we always mean its operator norm.

Since $\mathcal{F}(n, k)$ is a compact set the value

$$e_1(n, k) = inf\{d_1(V) : V \in \mathcal{F}(n, k)\}$$

is attained and we define the *1-erasure frames* to be the nonempty compact set $\mathcal{E}_1(n, k)$ of frames where this infimum is attained, i.e.,

$$\mathcal{E}_1(n, k) = \{V \in \mathcal{F}(n, k) : d_1(V) = e_1(n, k)\}.$$

Proceeding inductively, we now set, for $1 \leq m \leq n$,

$$e_m(n, k) = inf\{d_m(V) : V \in \mathcal{E}_{m-1}(n, k)\}$$

and define the *m-erasure frames* to be the nonempty compact subset $\mathcal{E}_m(n, k)$ of $\mathcal{E}_{m-1}(n, k)$ where this infimum is attained.

In this fashion, we obtain a decreasing family of frames and we wish to describe and construct the frames in these sets.

The results of [8] can be interpreted as characterizing $\mathcal{E}_1(n,k)$.

Proposition 2.1. [8] *The set $\mathcal{E}_1(n,k)$ coincides with the family of uniform (n,k)-frames, and consequently, $e_1(n,k) = k/n$.*

Proof. Given an (n,k)-frame $F = \{f_1, \dots, f_n\}$, if we regard the frame vectors as column vectors, then the analysis operator V is just the matrix whose p-th row is f_p^*. Given D in $\mathcal{D}_1$ which is 1 in the p-th entry, we have that

$$\|V^*DV\| = \|(\overline{v}_{p,i}v_{p,j})\| = \|f_pf_p^*\| = \|f_p\|^2,$$

where the last equality is easily seen by examining the action of the matrix $f_pf_p^*$ on a vector. Thus, we see that

$$d_1(V) = max\{\|f_p\|^2 : 1 \leq p \leq n\}.$$

Since $\sum_p \|f_p\|^2 = tr(VV^*) = k$, we have that $\|f_p\|^2 \geq k/n$ for some p. Hence $d_1(V)$ is clearly minimized when $\|f_p\|^2$ is the constant k/n independent of p. That is, when F is a uniform (n,k)-frame. □

We now turn our attention to finding frames that belong to $\mathcal{E}_2(n,k)$. By Proposition 2.1 these are the uniform (n,k)-frames which achieve the infimum of $e_2(n,k)$.

If D is in $\mathcal{D}_2$ and has a 1 in the i-th and j-th diagonal entries and V is the analysis operator for a uniform (n,k)-frame $F = \{f_1, \dots, f_n\}$, then $\|V^*DV\| = \|DVV^*D\| = k/n + |\langle f_i, f_j\rangle| = k/n(1 + |cos(\theta_{i,j})|)$ where $\theta_{i.j}$ is the angle between the i-th and j-the frame vector. Note that $|cos(\theta)| = |cos(\pi \pm \theta)|$. Consequently, $d_2(V) = k/n(1 + cos(\theta_F)$ where we set $\theta_F = max\{cos^{-1}(|cos(\theta_{i,j})|) : i \neq j\}$ and $0 \leq \theta_F \leq \pi/2$. Because $cos(\theta)$ is a decreasing function in this interval θ_F is attained where the angle between frame vectors is minimized $mod(\pi/2)$.

The following is immediate.

Proposition 2.2. *If $\Theta_{n,k} = inf\{\theta_F : F \in \mathcal{E}_1(n,k)\}$, then $e_2(n,k) = k/n(1 + cos(\Theta_{n,k})$ and an (n,k)-frame F is in $\mathcal{E}_2(n,k)$ if and only if it is a uniform frame and $\theta_F = \Theta_{n,k}$.*

The family of frames satisfying $\theta_F = \Theta_{n,k}$ is also introduced in [25], where they are called Grassmannian frames. Thus, Grassmannian frames are another term for the frames in $\mathcal{E}_2(n,k)$.

In the case when $k = 2$, it is possible to describe all frames in $\mathcal{E}_m(n,2)$.

Proposition 2.3. *For $m \geq 2$ and $n \geq 2$, every frame in $\mathcal{E}_m(n,2)$ is frame equivalent to the frame given by setting*

$$f_j = \sqrt{2/n}(cos(\frac{\pi j}{n}), sin(\frac{\pi j}{n})), j = 1, \dots n$$

and consequently $\Theta_{n,k} = \pi/n$.

Proof. Note that every frame is equivalent to one for which the second component is always non-negative. By the above proposition, a uniform frame that is in $\mathcal{E}_2(n,k)$ is one that makes the smallest angle between vectors as large as possible. The frame above is easily seen to achieve this minimum and to be unique up to frame equivalence.

Thus, $\mathcal{E}_2(n,2)$ consists of a single frame equivalence class and consequently, $\mathcal{E}_m(n,2) = \mathcal{E}_2(n,2)$ for all $m \geq 2$. □

In particular, we see that the above frame is the unique element of $\mathcal{E}_m(n,2)$, up to frame equivalence, and is the optimal $(n,2)$-frame for any number of erasures.

It is interesting to compare the angle $\Theta_{n,k}$, when $k = 3$ and n is arbitrary, to some of the angles computed for the best packings of lines into a sphere appearing in the work of Conway, Hardin and Sloane[12]. They find packings of n lines through the origin in $\mathbb{R}^3$ that maximize the minimal angle between lines, describe the packings and compute this minimal angle for $2 \leq n \leq 55$. For some values of n they are able to describe these angles and packings explicitly, while for other values they are only able to give numerical outcomes.

Since their packings are not constrained by our frame requirements, their angle for a particular n is always necessarily greater than or equal to our $\Theta_{n,3}$.

A natural question is whether or not one obtains a tight frame by choosing a unit vector from each of the lines in their optimal packing. Or equivalently, if by choosing a vector of length $\sqrt{3/n}$ from each of their lines one obtains a uniform $(n,3)$-frame. If one does obtain a uniform $(n,3)$-frame from one of their packings, then it is necessarily a frame in $\mathcal{E}_2(n,3)$ and in such a case their angle and $\Theta_{n,3}$ will be equal.

Fickus[15] shows that for various uniform solids, by choosing the unit vectors corresponding to the vertices, one obtains a tight frame. When these solids are symmetric under antipodal reflection, then one also obtains a tight frame by keeping one vector from each antipodal pair. In this fashion one obtains uniform tight frames for $n = 4, 6, 10$ and 30. For $n = 4, 6$ the frames considered by Fickus, correspond to the packings of [12] and so we know that these packings yield the optimal frames in this manner for $n = 4, 6$. However, when $n = 10, 30$ the solids considered by Fickus, differ from the solids corresponding to the optimal line packings of [12].

We will show that for some values of n the packings of [12] do not yield frames. But the set of integers n such that the optimal line packing does yield a frame in this manner, is not known.

Another closely related problem is Tammes' problem, which seeks the packing of n points on a sphere so as to maximize the minimum distance between any two points. Again, if the solution to Tammes' problem did yield a tight frame then that frame would be in $\mathcal{E}_2(n,3)$, but Fickus shows that generally, the solutions to Tammes' problem are not tight frames.

We have attempted to numerically compute $\Theta_{n,3}$ for the same values of n as is done in [12] and Table 1 displays the outcome of these calculations compared to the angles computed by [12]. For some values of n these angles appear to be equal and when it is known that this is the case, we indicate this in the notes column with a reference. For other values of n, these angles appear to be different and when we can prove that they are in fact different, we also have indicated that in the notes column.

These numerical calculations are only intended to be indicative of possible conclusions and we can make no claims about the accuracy of the outcomes of our calculations versus the actual values of $\Theta_{n,3}$. In fact, for some values of n, we consistently obtain numerical values for $\Theta_{n,3}$ that are slightly greater than the angle computed by [12], which is theoretically impossible.

There are two possible reasons for this difference, one is the inaccuracy of the calculations. The other is the fact that numerically, we are only finding frames that are nearly uniform and nearly normalized tight and the formula that we use for computing $\Theta(n, 3)$ assumes that the vectors are actually of equal length.

Using a compactness argument, as was pointed out to us by D. Hadwin, one can show that for every $\epsilon > 0$, there exists a $\delta > 0$ such that if a frame has frame bounds $C = 1 - \delta$ and $D = 1 + \delta$ and the lengths of the frame vectors differ by at most δ, then there is a uniform normalized tight frame, whose vectors are at distance at most ϵ from the original vectors. But we do not have any results that give us any control on the relative sizes of ϵ and δ.

If this ratio is extremely large, then computing $\Theta_{n,3}$ with much accuracy will be numerically difficult.

In cases where the actual value of their minimal angle is greater than $\Theta_{n,3}$, we have, by the above result, that their packing could not yield a frame. Not surprisingly, for most values of n their angle appears to be greater than $\Theta_{n,3}$. For $n = 3, 4, 6$, their minimum angle and $\Theta_{n,3}$ appear to be equal.

Thus, the numerical evidence suggests that their packings could yield frames in $\mathcal{E}_2(n, 3)$ for these values of n. Results in this paper will show that this is indeed the case for $n = 3, 4, 6$. For $n = 3$, it is clear since both are achieved by an orthonormal basis.

In the case of $n = 5$, [12] shows that one can pack 5 lines through the origin such that the angle between each pair of lines is equal to $cos^{-1}(\sqrt{1/3})$ which is about 63.4349. However, we will prove that for $n = 5$ the solution to the minimal line packing problem does not yield a uniform $(5, 3)$-frame.

Fickus[15] shows that the solution to Tammes' problem for $n = 5$ also is not a tight frame.

For this pair of integers we only have a numerical description of the frames in $\mathcal{E}_2(5,3)$ and no clear geometric understanding of these frames. In particular, we have been unable to determine whether or not all the frames in $\mathcal{E}_2(5,3)$ are frame equivalent.

There are many other values of n where the numerical calculations indicate that either the angles are equal or are very close and we currently have no proofs or intuition for these cases.

Definition 2.4. *We call F a* **2-uniform (n,k)-frame** *provided that F is a uniform (n,k)-frame and in addition $\|V^*DV\|$ is a constant for all D in $\mathcal{D}_2$.*

We will show later that, unlike uniform frames, 2-uniform frames do not exist for all values of k and n. However, we will prove that when they do exist then these are exactly the frames in $\mathcal{E}_2(n,k)$.

Theorem 2.5. *Let F be a uniform (n,k)-frame. Then F is 2-uniform if and only if $|\langle f_j, f_i\rangle| = c_{n,k}$ is constant for all $i \neq j$, where*

$$c_{n,k} = \sqrt{\frac{k(n-k)}{n^2(n-1)}}.$$

Proof. Fix $i \neq j$, let V be the analysis operator for F and let D be the diagonal matrix that is 1 in the (i,i) and (j,j) entries and is 0 elsewhere. Since $D^2 = D = D^*$, we have that

$$\|V^*DV\| = \|(DV)^*(DV)\| = \|DVV^*D\| = \| \begin{pmatrix} k/n & \langle f_i, f_j\rangle \\ \langle f_j, f_i\rangle & k/n \end{pmatrix} \|.$$

The norm of this 2×2 matrix is easily found to be $k/n + |\langle f_j, f_i\rangle|$ and thus F is 2-uniform if and only if $|\langle f_j, f_i\rangle|$ is constant, say c, for all $i \neq j$.

To see the final claim, use the fact that $P = VV^*$ satisfies $P = P^2$. Equating diagonal entries of P and P^2, yields the equation

$$k/n = (k/n)^2 + (n-1)c^2$$

which can be solved for c to yield the above formula for $c_{n,k}$.

□

The families of frames satisfying the latter condition in the above proposition have also been studied independently in [25], where they are called *equiangular frames.*

It is perhaps more instructive to state the above theorem in terms of the angles between the lines spanned by the frame vectors. Recalling that each frame vector has length $\sqrt{k/n}$, we have that $c_{n,k}\sqrt{n/k}$ is the cosine of the angle between the lines spanned by the frame vectors.

Corollary 2.6. *Let F be a uniform (n,k)-frame. Then F is 2-uniform if and only if the angle between the lines spanned by every pair of frame vectors is equal to*

$$\gamma_{n,k} = cos^{-1}(\sqrt{\frac{n-k}{k(n-1)}}).$$

Proposition 2.7. *Let natural numbers $k \leq n$ be given. If $F = \{f_1, \ldots, f_n\}$ is a uniform (n,k)-frame, then for each i there exists $j \neq i$ such that $|\langle f_j, f_i \rangle| \geq c_{n,k}$. Consequently, if V denotes the analysis operator of F, then $d_2(V) \geq k/n + c_{n,k}$.*

Proof. Let $P = (p_{i,j}) = VV^*$ denote the correlation matrix of F. Using the fact that $P^2 = P$ and equating the (i,i)-th entry yields $\sum_{j=1}^n |p_{i,j}|^2 = (k/n)$ and hence,

$$\sum_{\substack{j=1 \\ j\neq i}}^n |p_{i,j}|^2 = (k/n) - |p_{i,i}|^2 = (k/n) - (k/n)^2 = \frac{k(n-k)}{n^2}.$$

Since there are $(n-1)$ terms in the above sum, at least one term must be larger than $\frac{k(n-k)}{(n-1)n^2} = c_{n,k}^2$, and the first result follows.

The second claim follows from the formula for $\|V^*DV\|$ for any D in $\mathcal{D}_2$ obtained in the proof of Proposition 2.5. □

Theorem 2.8. *Let natural numbers $k \leq n$ be given. If there exists a 2-uniform (n,k)-frame, then every frame in $\mathcal{E}_m(n,k)$ is 2-uniform for $2 \leq m$*

and $e_2(n,k) = k/n + c_{n,k}$ *and* $\Theta_{n,k} = \gamma_{n,k}$. *If there does not exist a 2-uniform* (n,k)*-frame, then necessarily* $e_2(n,k) > k/n + c_{n,k}$ *and* $\Theta_{n,k} < \gamma_{n,k}$.

Proof. The first statement follows from Proposition 2.7. To see the second statement, note that by compactness there must exist a uniform (n,k)-frame F with analysis operator V such that $e_2(n,k) = d_2(V)$. If $e_2(n,k) = k/n + c_{n,k}$, then the proof of the above proposition shows that for all $j \neq i$, we would have that

$$|\langle f_j, f_i \rangle| = |p_{i,j}| = c_{n,k},$$

which implies that F is 2-uniform. □

In the case of $n = 5$, [12] shows that one can pack 5 lines through the origin such that the angle between each pair of lines is equal to 63.4349. However, if this was a $(5,3)$-frame, then it would be 2-uniform and consequently, the angle between pairs would necessarily be $\gamma_{5,3} = cos^{-1}(\sqrt{1/6})$ which is approximately equal to 65.90515745. Thus, we can conclude that for $n = 5$ the solution to the minimal line packing problem does not yield a uniform $(5,3)$-frame.

Although we have been able to produce actual $(5,3)$-frames that agree with the computed numerical minimal value, we do not have a clear geometric understanding of these frames. In particular, we do not know if all the frames in $\mathcal{E}_2(5,3)$ are frame equivalent.

In the next section we will discuss existence and construction of 2-uniform frames and we will show that, in the real case, for many possible values of (n,k), there do not exist any 2-uniform frames. In the real case when there do exist 2-uniform frames, we will show that there are at most finitely many such frames and hence the problem of determining optimal frames in our sense, i.e., frames in $\mathcal{E}_m(n,k)$, is reduced to the problem of determining which one of these finitely many frames is optimal.

3. Existence and construction of 2-uniform frames

In this section we study the problems of the existence and construction of 2-uniform frames. Since the inner products are of constant modulus for a 2-uniform (n,k)-frame, it is fairly easy to see that for a given value of k, the integer n is bounded and so 2-uniform frames can not exist for all pairs (n,k). In particular, in the real case, when $k=2$, then we can have at most 3 vectors whose inner products are of constant modulus. Proceeding inductively, one can get a crude upper bound of $n \leq 2^k - 1$. However, much better bounds are known, in fact, $n \leq k^2$ in the complex case and $n \leq k(k+1)/2$ in the real case, see [25].

Given a 2-uniform (n,k)-frame $F = \{f_1, \ldots, f_n\}$ the correlation matrix is a self-adjoint rank k projection that can be written in the form $P = VV^* = aI + cQ$ where $a = k/n, c = c_{n,k}$ is given by the formula derived in the last section and $Q = (q_{i,j})$ is a self-adjoint matrix satisfying $q_{i,i} = 0$ for all i and for $i \neq j, |q_{i,j}| = 1$. We shall derive further properties that the matrix Q must satisfy and then use solutions of these equations to generate 2-uniform frames and use the impossibility of solution to these equations to rule out the possibility of the existence of 2-uniform frames for certain pairs (n,k).

Definition 3.1. *If F is a 2-uniform (n,k)-frame, then we call the $n \times n$ self-adjoint matrix Q obtained above the* **signature matrix** *of F.*

The fact that in the real case Q must be a matrix of 0's,1's and -1's satisfying an algebraic equation shows that given a 2-uniform (n,k)-frame there are only finitely many possibilities for its Grammian matrix. Consequently up to equivalence there can be only finitely many 2-uniform (n,k)-frames for each pair (n,k).

Proposition 3.2. *If Q is the signature matrix of a 2-uniform (n,k)-frame, then*

$$Q^2 = (n-1)I + \mu_{n,k} Q$$

with

$$\mu_{n,k} = (n-2k)\sqrt{\frac{n-1}{k(n-k)}}.$$

If, in addition Q is the signature matrix of a real 2-uniform (n,k)-frame, then $\mu_{n,k}$ is an integer.

Proof. This result follows from using the identity $P^2 = P$ and the fact that when the frame is real all the entries of Q and Q^2 must be integers. □

The fact that $\mu_{n,k}$ must be an integer in the real case rules out the possibility of the existence of real 2-uniform frames for many values of (n,k). For example, in this manner we can see that there is no real 2-uniform $(7,3)$-frame, even though this pair of values satisfies the inequality of [25] and of Proposition 3.1. Later we shall construct real 2-uniform $(6,3)$-frames.

We now prove the converse of the above proposition.

Theorem 3.3. *Let Q be a self-adjoint $n \times n$ matrix Q with $q_{i,i} = 0$ for all i and $|q_{i,j}| = 1$ for all $i \neq j$. Then the following are equivalent:*

i) *Q is the signature matrix of a 2-uniform (n,k)-frame,*

ii)
$$Q^2 = (n-1)I + \mu Q$$
for some necessarily real number μ,

iii) *Q has exactly two eigenvalues, $\rho_1 \geq \rho_2$.*

When any of these equivalent conditions hold then the parameters k, μ, ρ_1, ρ_2 are related by the equation given in Proposition 3.2 and by the equations,

$$\mu = \rho_1 + \rho_2, 1 - \rho_1\rho_2 = n,$$
$$k = n/2 - \frac{\mu n/2}{\sqrt{4(n-1)+\mu^2}} = \frac{-n\rho_2}{\rho_1 - \rho_2}.$$

In particular, solutions of these equations can only exist for real numbers μ such that the formula for k is an integer.

Proof. We have already seen that i) implies ii). To see that ii) implies i), it is sufficient to show that for appropriately chosen values of a and c, the self-adjoint matrix $P = aI + cQ$ satisfies $P^2 = P$ for then P will be the matrix of a projection of integer rank and by factoring $P = VV^*$ we will obtain the desired frame.

It is now readily checked that if we set

$$2a = 1 - \frac{m}{\sqrt{4(n-1)+m^2}}$$

and

$$c^2 = \frac{a-a^2}{n-1} = \frac{1}{4(n-1)+m^2}$$

then $P^2 = P$.

Note that ii) implies iii), because if ii) holds then Q satisfies a second degree polynomial and so has at most two eigenvalues.

Finally, to see that iii) implies ii). Note that iii) implies that

$$Q^2 = (\rho_1 + \rho_2)Q - (\rho_1\rho_2)I.$$

However, since the diagonal entries of Q are all 0 and the diagonal entries of Q^2 are all (n-1), we necessarily have that $\rho_1\rho_2 = 1 - n$ and so ii) holds. □

The above theorem makes it possible to construct 2-uniform frames.

Note that if Q is a signature matrix for a 2-uniform (n, k)-frame then $-Q$ is also a signature matrix for a 2-uniform $(n, n-k)$-frame. It is easily seen that if P is the corresponding projection for Q, then the projection corresponding to $-Q$ is $I - P$ which is the orthocomplement of the first subspace.

This observation leads to some improvement on the bound from [25].

Proposition 3.4. *If there exists a 2-uniform (n, k)-frame, then necessarily, $n \leq min\{k^2, (n-k)^2\}$ in the complex case and $n \leq min\{k(k+1)/2, (n-k)(n-k+1)/2\}$ in the real case.*

This pair of inequalities gives upper and lower bounds for n and k.

These inequalities rule out the existence of 2-uniform frames for many pairs (n, k). For example, 2-uniform $(k+2, k)$-frames could only exist for

$k = 1$ in the real case and $k = 2$ in the complex case. Thus, in particular we have proven that there are no real or complex 2-uniform $(5, 3)$-frames.

Thus, in particular, there can be no real 2-uniform $(4, 2)$-frame. This can also be checked by showing directly that there are no possible 4×4 real signature matrices. Note that in the complex case, Proposition 3.1 allows for the existence of a complex 2-uniform $(4, 2)$-frame and we shall show below that one in fact exists.

We now use the existence of signature matrices to construct 2-uniform frames.

Example 3.5. *The codimension 1 case.*

Let J denote the $n \times n$ matrix all of whose entries are 1. Then $Q = J - I$ satisfies $Q^2 = J^2 - 2J + I = (n - 2)J + I = (n - 1)I + (n - 2)Q$ and so by our above formulas $\mu = (n - 2), k = 1$ and so yields the rather uninteresting 2-uniform frame for $\mathbb{F}^1$.

However, $-Q = I - J$ is also a signature matrix with $\mu = (2 - n), k = (n-1)$, which shows that for each k there exists a 2-uniform $(k+1, k)$-frame.

This frame is described in detail in [8] and is in fact the only real uniform $(k + 1, k)$-frame, up to some natural equivalence.

Thus, $\mathcal{E}_1(k + 1, k) = \mathcal{E}_m(k + 1, k)$ consists of the 2-uniform frames that are frame equivalent to this frame.

Thus, we have that $\Theta_{k+1,k} = \gamma_{k+1,k} = cos^{-1}(\sqrt{1/k})$. In particular, when $k = 3$, we find that the actual value of $\Theta_{4,3}$ agrees with the actual value of the angle computed by [12] and the optimal packing that they describe yields a 2-uniform $(4, 3)$-frame.

Example 3.6. *Conference Matrices.*

A real $n \times n$ matrix C with $c_{i,i} = 0$ and $c_{i,j} = \pm 1$ for $i \neq j$ is called a *conference matrix* [13] provided $C^*C = (n - 1)I$.

Thus, every symmetric conference matrix is a signature matrix with $\mu = 0$ and $k = n/2$. So, in particular such matrices must be of even size and they yield real 2-uniform $(2k, k)$-frames, for certain values of k.

If $C = -C^t$ is a skew-symmetric conference matrix, then setting $Q = iC$ yields a complex 2-uniform $(2k, k)$-frame.

The idea of using conference matrices to construct frames of this type originates in [25].

The smallest example of a symmetric conference matrix is given by the 6×6 matrix,

$$\begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & -1 & -1 & 1 & 1 \\ 1 & -1 & 0 & 1 & -1 & 1 \\ 1 & -1 & 1 & 0 & 1 & -1 \\ 1 & 1 & -1 & 1 & 0 & -1 \\ 1 & 1 & 1 & -1 & -1 & 0 \end{pmatrix},$$

which gives rise to a real 2-uniform $(6, 3)$-frame.

Thus, we have that $\Theta_{6,3} = \gamma_{6,3} = cos^{-1}(\sqrt{1/3})$ and the line packing described in [12] yields this 2-uniform $(6, 3)$-frame.

With a little work, one can show that up to conjugation by a unitary that is the product of a permutation and a diagonal unitary, the above 6×6 matrix is the unique symmetric conference matrix of this size.

Thus, $\mathcal{E}_2(6, 3) = \mathcal{E}_m(6, 3)$ for $m \geq 2$, consists of the frame equivalence class of this frame.

The smallest examples of skew symmetric conference matrices are given by the 4×4 matrices,

$$\begin{pmatrix} 0 & 1 & 1 & 1 \\ -1 & 0 & -1 & 1 \\ -1 & 1 & 0 & -1 \\ -1 & -1 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 1 & 1 \\ -1 & 0 & 1 & -1 \\ -1 & -1 & 0 & 1 \\ -1 & 1 & -1 & 0 \end{pmatrix}$$

which give rise to complex 2-uniform $(4, 2)$-frames.

Recall that we showed earlier that there does not exist any real 2-uniform $(4, 2)$-frame.

The 2-uniform frames arising from the two skew symmetric conference matrices given above are unitarily equivalent via a unitary matrix that is a product of a permutation and a diagonal unitary and hence these two different matrices really give rise to only one frame equivalence class.

Example 3.7. *Complex Examples.*

The following 4×4 complex signature matrices satisfy $Q^2 = 3I$ and give rise to complex 2-uniform $(4, 2)$-frames,

$$Q_1 = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & -i & i \\ 1 & i & 0 & -i \\ 1 & -i & i & 0 \end{pmatrix}, Q_2 = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & i & -i \\ 1 & -i & 0 & i \\ 1 & i & -i & 0 \end{pmatrix}.$$

These two matrices are unitarily equivalent via a diagonal unitary to iC_1 and iC_2 where C_1 and C_2 are the skew symmetric conference matrices appearing in the above example and so the frames arising from them are all equivalent to the frames of the previous example.

In fact it is possible to show that any 4×4 signature matrix satisfying $Q^2 = 3I$ is unitarily equivalent to the above example via a unitary that is the product of a permutation and a diagonal unitary and hence all 2-uniform $(4, 2)$-frames are frame equivalent.

Example 3.8. *Hadamard Matrices.*

A real $n \times n$ matrix H is called a *Hadamard matrix* [13] provided that $h_{i,j} = \pm 1$ and $H^*H = nI$. If $H = H^*$ is a symmetric Hadamard matrix and in addition, $h_{i,i} = 1$ for all i, then $Q = H - I$ is a signature matrix with $\mu = -2$ and $k = \frac{n+\sqrt{n}}{2}$.

Two such examples are given by the matrices,

$$\begin{pmatrix} 1 & -1 & -1 & -1 \\ -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}.$$

However, the frames arising from these two matrices can be shown to be equivalent.

Given two such Hadamard matrices their Kronecker tensor product gives rise to another such Hadamard matrix. Thus, using the above matrices, one obtains 2-uniform $(4^j, 2^{2j-1} \pm 2^{j-1})$-frames for each integer j.

The formula for k shows that such Hadamard matrices can only exist when n is a perfect square. Solving for n in terms of k, we find that 2-uniform frames can arise in this fashion only when $8k + 1$ is a perfect square.

Similarly, $-Q = I - H$ is a signature matrix for $\mu = 2$ and $k_1 = n - k = \frac{n-\sqrt{n}}{2}$ which again implies that $8k_1 + 1$ is a perfect square.

Thus to construct a real 2-uniform (n, k)-frame by these means, one finds that necessarily $n, 8k + 1$ and $8(n - k) + 1$ need to be perfect squares. We have seen that examples exist for $n = 4^j$.

The next smallest possible value is $n = 36$. Bussemaker and Seidel[3] show that there are 92 symmetric Hadamard matrices of this size and so these matrices yield real 2-uniform $(36, k)$-frames for $k = 15, 21$.

In algebra, two Hadamard matrices H_1 and H_2 are usually considered equivalent if there exist U and V which are products of permutations and diagonal matrices of $\pm 1's$ such that $UH_1V = H_2$. For many values of n a representative from each such equivalence class of Hadamard matrices is available. For this coarse equivalence relation, there exist only 2 inequivalent Hadamard matrices of order 36. Most sources that exhibit all representative Hadamard matrices of a certain order, refer to this coarse equivalence.

However, if two Hadamard matrices H_1 and H_2 are symmetric so that both give rise to frames, then those frames are frame equivalent if and only if $UH_1U^* = H_2$ for some U as above. This equivalence relation is finer than the usual equivalence relation and is called *switching* in the work of Seidel, et al. Thus it is possible that a single representative Hadamard matrix could give rise to many inequivalent frames. In fact, [3] show that of the 92 symmetric Hadamard matrices of order 36, only 2 are equivalent via conjugation by diagonal unitaries.

Thus, these 91, inequivalent under switching, Hadamard matrices give rise to 91 inequivalent 2-uniform $(36, 15)$-frames and 91 inequivalent 2-uniform $(36, 21)$-frames.

To obtain the optimal $(36, 15)$-frame for $m > 2$ erasures, one needs to compute the numbers $d_m(V)$ for the analysis operators of each of these 91 frames. We have been able to show that $d_3(V)$ is the same value for all of these frames and so $\mathcal{E}_2(36, 15) = \mathcal{E}_3(36, 15)$. However, we believe that $d_4(V)$ is not constant for these 91 frames, so that some of these Hadamard matrices give rise to frames that behave better for 4 erasures.

Example 3.9. *A 2-uniform (28,7)-frame.*

The smallest value of (n,k) for which μ is an integer, that is not covered by any of the above cases is $n = 28, k = 7$ and $\mu = 6$. In [19] a 28 by 28 matrix Q satisfying $Q^2 = 27I + 6Q$ is exhibited and thus one obtains a 2-uniform $(28, 7)$-frame. This signature matrix is obtained from the adjacency matrix of the first of the strongly regular graphs on 28 vertices appearing in [24] by replacing its' standard adjacency matrix by its' Seidel adjacency matrix.

Given a graph G on n vertices, the *Seidel adjacency matrix of G* is defined to be the $n \times n$ matrix $A = (a_{i,j})$ where $a_{i,j}$ is defined to be -1 when i and j are adjacent, $+1$ when i and j are not adjacent, and 0 when $i = j$. Two graphs on n vertices are called *switching equivalent* exactly when their Seidel adjacency matrices are unitarily equivalent via a unitary that is the product of a permutation and a diagonal matrix of $\pm$ 1's.

Note that two real 2-uniform frames are frame equivalent exactly when their signature matrices give rise to switching equivalent graphs.

A *two-graph* (Ω, Δ) is a pair consisting of a vertex set Ω and a collection Δ of three element subsets of Ω such that every four element subset of Ω contains an even number of the sets from Δ. A two-graph is regular, provided that every two element subset of Ω is contained in the same number, α, of sets in Δ.

Given n, Seidel[23] exhibits a one-to-one correspondence between the two-graphs on the set of n elements and the switching equivalence classes of graphs on n elements and gives a concrete means, given the two-graph, to construct a graph from the corresponding switching class.

Thus, a two-graph can be regarded as a switching equivalence class of ordinary graphs.

In [25], it was noted that signature matrices of real 2-uniform frames are always Seidel adjacency matrices of regular two-graphs. Theorem 3.3 allows us to more fully summarize this connection.

Theorem 3.10. *An $n \times n$ matrix Q is the signature matrix of a real 2-uniform (n, k)-frame if and only if it is the Seidel adjacency matrix of a graph on n vertices whose switching equivalence class is a regular two-graph on n vertices with parameter α. This relationship defines a one-to-one correspondence between frame equivalence classes of real 2-uniform frames and regular two-graphs.*

Proof. Seidel[23], proves that a two-graph is regular if and only if the graphs in the switching class that it determines all have 2 eigenvalues. But by Theorem 3.3, these are exactly the adjacency graphs of signature matrices. □

The relationship between the parameter α and earlier parameters is given by the equations,

$$-2\alpha = (1 + \rho_1)(1 + \rho_2) = 2 + \mu + n.$$

Thus, by the above theorem every regular two-graph produces a real 2-uniform frame. For a given n these could just be the trivial, known examples corresponding to $k = n - 1, 1$. In [23] many of the known regular two-graphs are listed and it is elementary to use the formulas given above to determine the pairs (n, k) for which they yield a real 2-uniform frame.

In particular, the two-graph $\Omega^-(6,2)$ yields a 2-uniform $(28,7)$-frame, but we have not determined whether or not it is frame equivalent to the frame generated by the signature matrix generated by Holmes[19].

4. Spectral Frames

For the linear coding theory viewpoint, all one really needs is a one-to-one linear transformation $A : \mathbb{F}^k \to \mathbb{F}^n$ that plays the role of the encoding operator and a left inverse $B : \mathbb{F}^n \to \mathbb{F}^k$ that plays the role of the decoding operator. We will call such a pair of matrices (A, B) an *(n,k)-code.* In the language of frame theory, the columns of A^* are the *frame vectors,* and we denote these by $\{a_1^*, \dots, a_n^*\}$ and the columns of B would be called the *dual frame vectors* and we denote these by $\{b_1, \dots, b_n\}$.

Note that the $n \times n$ matrix $AB = (a_i b_j)$ is an idempotent matrix of rank k. Conversely, given an $n \times n$ idempotent matrix E of rank k, it is possible to factor $E = AB$ with $BA = I_k$.

We begin this section by returning to the topic of the second section in this more general setting. If we tried to minimize the norms of the error operators in this setting, we would quickly find ourselves back in the situation of section 2. Namely, dealing with A an isometry and $B = A^*$, so we would have normalized tight frames. We believe that in this setting it is more meaningful to minimize the spectral radii of the error operators.

As in section 2, if we assume that m components of our vector are lost in transmission, but still use the left inverse B to attempt to reconstruct the transmitted vector then the error will be BDA where D is a diagonal matrix with m 1's and $n - m$ 0's on its diagonal and we are interested in choosing pairs (A, B) as above which somehow minimize $r(BDA)$, where $r(X)$ denotes the spectral radius of a matrix X.

We let $\mathcal{C}_0(n,k) = \{(A,B) : BA = I_k\}$ where A is an $n \times k$ matrix and B is a $k \times n$ matrix and let $\mathcal{D}_m$ denote the set of $n \times n$ diagonal matrices with

m 1's and $n-m$ 0's. We set

$$r_m(A,B) = min\{r(BDA) : D \in \mathcal{D}_m\}.$$

Finally, we inductively define

$$s_m(n,k) = inf\{r_m(A,B) : (A,B) \in \mathcal{C}_{m-1}(n,k)\}$$

and

$$\mathcal{C}_m(n,k) = \{(A,B) \in \mathcal{C}_{m-1}(n,k) : r_m(A,B) = s_m(n,k)\}.$$

Of course, if the above infimum defining $s_m(n,k)$ is not attained, then $\mathcal{C}_m(n,k)$ will be empty.

The first proposition shows that the set $\mathcal{C}_1(n,k)$, in many ways, mimics the uniform frames.

Proposition 4.1. *Let $0 < k \leq n$, be integers. Then $s_1(n,k) = k/n$ and $\mathcal{C}_1(n,k) = \{(A,B) \in \mathcal{C}_0(n,k) : a_i b_i = k/n, 1 \leq i \leq n\}$.*

Proof. Recall that $r(XY) = r(YX)$ and hence $r(BDA) = r(DABD)$. If $D \in \mathcal{D}_1$ is 1 in the i-th diagonal entry, then $r(DABD) = |a_i b_i|$.

Since $tr(AB) = tr(BA) = k$, we see that the infimum defining $s_1(n,k)$ is attained by any pair (A,B) satisfying $a_i b_i = k/n$ for all i. □

We call an (n,k)-code (A,B) *uniform* if $a_i b_i$ is constant in which case it must be equal to k/n and we call it a *2-uniform* (n,k)-code provided that it is uniform and $r(BDA)$ is constant as D varies over all diagonal matrices in $\mathcal{D}_2$.

Theorem 4.2. *Let $0 < k \leq n$, be integers. A uniform (n,k)-code is 2-uniform if and only if $(a_i b_j)(a_j b_i) = c_{n,k}^2$ where $c_{n,k} = \sqrt{\frac{k(n-k)}{n^2(n-1)}}$. If there exists a 2-uniform (n,k)-code, then every code in $\mathcal{C}_m(n,k)$ is 2-uniform for $2 \leq m$ and $s_2(n,k) = k/n + c_{n,k}$.*

Proof. The first statement comes from observing that the spectral radius of the 2×2 matrix $DABD$ is $k/n + (a_i b_j)(a_j b_i)$, provided that the latter quantity is non-negative, and using the fact that AB is an idempotent, as in the proof of Theorem 2.3.

The second statement follows as in the proof of Theorem 2.6. □

If (A, B) is a 2-uniform (n, k)-code, then we may write the idempotent $AB = \frac{k}{n}I + c_{n,k}Q$ where $Q = (q_{i,j})$ satisfies $q_{i,i} = 0$ and $q_{i,j}q_{j,i} = 1$ for all $i \neq j$.

We shall call a matrix Q that satisfies these last two conditions a *generalized signature matrix.*

The following results are the analogues of Proposition 3.2 and Theorem 3.3.

Proposition 4.3. *Let $0 < k \leq n$ be integers. If (A, B) is a 2-uniform (n, k)-code and Q is its generalized signature matrix, then*

$$Q^2 = (n-1)I + \mu_{n,k}Q$$

where $\mu_{n,k} = (n-2k)\sqrt{\frac{n-1}{k(n-k)}}$. Conversely, if an $n \times n$ generalized signature matrix Q satisfies an equation of the form

$$Q^2 = (n-1)I + \mu Q$$

with $\mu^2 \neq -4(n-1)$, then Q is the generalized signature matrix of a 2-uniform (n, k)-code (A, B) with $k = n/2 - \frac{\mu n/2}{\sqrt{4(n-1)+\mu^2}}$ and $AB = \frac{k}{n}I + c_{n,k}Q$.

We have so far been unable to construct a generalized signature matrix that is not equivalent to a signature matrix and we do not know if the analogue of Theorem 3.3iii) holds.

We also have not been able to rule out the possibility that a generalized signature matrix exists that satisfies an equation of the form $Q^2 = (n-1)I + \mu Q$ with $\mu^2 = -4(n-1)$.

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TABLE 1. Optimal Angle for Frames in $\mathbb{R}^3$

No. of vectors N	$\Theta_{n,3}$	CHS Min Angle	Notes
4	70.5288	70.5288	equality shown [15]
5	57.3647	63.4349	inequality shown
6	63.4349	63.4349	equality shown [15]
7	54.7356	54.7356	
8	47.8467	49.63.99	
9	46.2666	47.9821	
10	46.6746	46.6746	
11	43.9060	44.4031	
12	41.8820	41.8820	
13	39.3442	39.8131	
14	38.1073	38.6824	
15	37.7612	38.1349	
16	37.3774	37.3774	
17	34.8364	35.2353	
18	34.2482	34.4088	
19	32.7366	33.2115	
20	32.6248	32.7071	
21	32.0227	32.2161	
22	31.9083	31.8963	
23	30.2325	30.5062	
24	30.0793	30.1628	
25	29.0723	29.2486	
26	28.4608	28.7126	
27	28.1378	28.2495	
28	27.6945	27.8473	
29	26.9727	27.5244	
30	26.5651	26.9983	
31	26.5053	26.4987	
32	26.0006	25.9497	
33	25.7908	25.5748	
34	24.9513	25.2567	
35	24.7417	24.8702	
36	24.3927	24.5758	
37	24.1935	24.2859	
38	24.0690	24.0886	
39	23.8466	23.8433	
40	23.2289	23.3293	
41	22.9488	22.9915	
42	22.5771	22.7075	
43	22.5111	22.5383	
44	22.1133	22.2012	
45	22.0326	22.0481	
46	21.7749	21.8426	
47	21.6640	21.6609	
48	21.4472	21.4663	
49	21.1602	21.1610	
50	20.9443	20.8922	
51	20.6968	20.6903	