ANALYSIS PRELIMINARY EXAM, PART I, FALL 2004

NAME:

There are two parts to this exam. On this side are true-false questions and on the reverse side you will find problems requiring proofs.

True-False. You will receive +5 for every correct answer, -5 for every incorrect answer and 0 for questions that are left blank. Write T if the statement is true, F if the statement is false and leave blank any questions when you are too unsure of the answer.

We shall let λ denote Lebesgue measure on $\mathbb{R}$.

1. The set $\mathbb{Q}$ of rational numbers is a dense subset of $\mathbb{R}$ of Lebesgue measure 0.

2. If E is a Lebesgue measurable subset of $\mathbb{R}$ then there is a Borel set $B \subset E$ such that $E \backslash B$ has Lebesque measure 0.

3. If $\{f_n\}$ is a sequence of Lebesgue measurable functions on $\mathbb{R}$, then the set of all x's such that $lim_n f_n(x)$ exists is a Lebesgue measurable set.

4. If f(x) is a Borel measurable function and g(x) is a measurable function, then g(f(x)) is a measurable function.

5. If f is of bounded variation on [a,b], then there exists an integrable function g such that $f(x) = f(a) + \int_{[a,x]} g d\lambda$ for $a \leq x \leq b$.

6. If f is absolutely continuous on [a,b], then f is differentiable almost everywhere.

7. If $\{f_n\}$ is a sequence of integrable functions on [0,1] and f is another integrable function such that $\int_{[0,1]} |f_n - f| d\lambda \to 0$ as $n \to +\infty$, then for almost every $x, lim_{n\to+\infty} f_n(x) = f(x)$.

8. If μ is a Borel measure on $\mathbb{R}$ then there exists Borel measures μ_1 and μ_2 such that $\mu = \mu_1 + \mu_2$, there exists an integrable function on $\mathbb{R}$ such that $\mu_1(B) = \int_B g d\lambda$, for every Borel set B, and there exists a Borel set B_0 such that $\lambda(B_0) = 0$ and $\mu_2(\mathbb{R}\backslash B_0) = 0$.

9. Let X be a real Banach space and let $L_n : X \to \mathbb{R}$ be a sequence of bounded linear functionals. If $lim_{n\to\infty} L_n(x)$ exists for every $x \in X$ and we define $L(x)$ to be equal to this limit, then L is also a bounded linear functional.

10. Let X be a real normed space and let $x \in X, x \neq 0$ then there exists a bounded linear functional, $L : X \to \mathbb{C}$ such that $\|L\| = \|x\|^{-1}$ and $L(x) = 1$.

Analysis Preliminary Exam, Part II, Fall 2004

We use λ to denote Lebesgue measure on $\mathbb{R}$.

1. Let $E \subset \mathbb{R}$ and let $a > 0$ be fixed, we set $aE = \{ax : x \in E\}$. Prove that $\lambda^*(aE) = a\lambda^*(E)$, where λ^* denotes Lebesgue outer measure. Prove that if E is measurable then aE is measurable.

2. Let $\mathcal{C}$ denote the σ-algebra of subsets of $\mathbb{R}^2$ generated by the sets of the form $[a, b) \times [c, d)$, with $-\infty < a < b < +\infty$, $-\infty < c < d < +\infty$. Prove that $\mathcal{C}$ coincides with the σ-algebra of Borel subsets of $\mathbb{R}^2$..

3. Let $C([0, 1])$ denote the Banach space of continuous functions on [0,1] equipped with the sup norm. Let $L : C([0, 1]) \to \mathbb{R}$ be the linear functional defined by

$$L(f) = -f(0) + \int_0^1 f(x^2)dx.$$

Prove that L is a bounded linear functional, describe the Borel measure such that $L(f) = \int_{[0,1]} f d\mu$. Give the decomposition of the signed measure μ as a difference of positive measures and compute $\|L\|$.

4. Let $\{f_n\}$ be a sequence of non-negative Lebesgue measurable functions on [0,1], and set $f(x) = limsup_n f_n(x)$. Either prove $\int_{[0,1]} f d\lambda = limsup_n \int_{[0,1]} f_n d\lambda$ or give a counterexample.

5. Recall that a function $f : [0, 1] \to \mathbb{R}$ is said to be of *bounded variation* if $V(f) = sup\{\sum_{i=0}^n |f(x_{i+1} - f(x_i)| : 0 = x_1 < x_2 < \ldots x_n = 1\}$ is finite, where the supremum is taken over all partitions of[0,1]. Prove that $\|f\| = |f(0)| + V(f)$ is a norm on the set $BV([0, 1])$ of functions of bounded variation and that it is a Banach space in this norm.