MATH 2331 FINAL EXAM NAME_______________

Review sheet ID #_______________

NO CALCULATORS!

1. Let $f(x)=\det\begin{bmatrix}2&5&0&0\\1&x&3&2\\0&6&5&0\\0&8&0&7\end{bmatrix}$. Find $f'(x)$. $\boxed{f'(x)=70}$. 12 pts

2. A rank one 3x3 symmetric matrix has column space containing the vector (1,2,3). Find a basis and the dimension of the null space. $\boxed{\text{Basis: } \{(-2,1,0),\ (-3,0,1)\} \text{ Dimension: } 2}$ 12 pts

3. a. Find the eigenvalues and eigenvectors of each of these matrices. Identify which are invertible and/or diagonalizable.

$A=\begin{bmatrix}0&1\\-1&0\end{bmatrix}$ $\boxed{\lambda_1=i,\ \mathbf{v}_1=(1,i),\ \lambda_2=-i,\ \mathbf{v}_2=(1,-i) \text{ Invertible,}}$

$\boxed{\text{Diagonalizable}}$ 6 pts

$B=\begin{bmatrix}2&3\\-1&-2\end{bmatrix}$ $\boxed{\lambda_1=1,\ \mathbf{v}_1=(-3,1), \lambda_2=-1,\ \mathbf{v}_2=(-1,1) \text{ Invertible,}}$

$\boxed{\text{Diagonalizable}}$ 6 pts

$C=\begin{bmatrix}2&5\\-1&-2\end{bmatrix}$ $\boxed{\lambda_1=i,\ \mathbf{v}_1=(-2-i,1),\ \lambda_2=-i,\ \mathbf{v}_2=(-2+i,1) \text{ Invertible,}}$

$\boxed{\text{Diagonalizable}}$

$D=\begin{bmatrix}0&1\\0&0\end{bmatrix}$ $\boxed{\lambda_1=\lambda_2=0,\ \mathbf{v}=(1,0) \text{ Neither invertible}}$

$\boxed{\text{nor diagonalizable.}}$ 6 pts

$E=\begin{bmatrix}0&1\\1&0\end{bmatrix}$ $\boxed{\lambda_1=1,\ \mathbf{v}_1=(-1,1), \lambda_2=-1,\ \mathbf{v}_2=(1,1) \text{ Invertible,}}$

$\boxed{\text{Diagonalizable}}$

$F=\begin{bmatrix}2&-1\\4&-2\end{bmatrix}$ $\boxed{\lambda_1=\lambda_2=0,\ \mathbf{v}=(1,2) \text{ Neither invertible}}$

$\boxed{\text{nor diagonalizable.}}$ 6 pts

b. Organize the matrices A-F in disjoint sets that of similar matrices.

$\boxed{AC}$ $\boxed{BE}$ $\boxed{DF}$ 6 pts

4. a. Find the determinant of this N-shaped matrix:

$$N = \begin{bmatrix} 1 & 0 & 0 & 4 \\ 2 & 1 & 0 & 3 \\ 3 & 0 & 1 & 2 \\ 4 & 0 & 0 & 1 \end{bmatrix}$$

-15

10 pts

b. What is the rank of N-I? Find all four eigenvalues of N.

N-I has rank 2. Eigenvalues 1, 1, -3, 5

5. For what vectors $\mathbf{b}$ does the system $\mathbf{Ax} = \mathbf{b}$ have a solution, if

$$\mathbf{A} = \begin{bmatrix} 1 & 3 & 1 \\ 0 & -1 & -1 \\ 2 & 4 & 0 \end{bmatrix}$$

? Find an equation for $\mathbf{b}$: $c_1 b_1 + c_2 b_2 + \ldots + c_n b_n = 0$

$2b_1 + 2b_2 - b_3 = 0$

12 pts

6. a. Suppose $\mathbf{q}_1, \mathbf{q}_2, \mathbf{a}_3$ are linearly independent vectors. $\mathbf{q}_1$ and $\mathbf{q}_2$ are already orthonormal. Give a formula for a third orthonormal vector $\mathbf{q}_3$ as a linear combination of $\mathbf{q}_1, \mathbf{q}_2, \mathbf{a}_3$.

$$\mathbf{q}_3 = \frac{\left(\mathbf{a}_3 - \mathbf{q}_1\mathbf{q}_1^{\mathbf{T}}\mathbf{a}_3 - \mathbf{q}_2\mathbf{q}_2^{\mathbf{T}}\mathbf{a}_3\right)}{\left\|\mathbf{a}_3 - \mathbf{q}_1\mathbf{q}_1^{\mathbf{T}}\mathbf{a}_3 - \mathbf{q}_2\mathbf{q}_2^{\mathbf{T}}\mathbf{a}_3\right\|}$$

10 pts

b. Find the vector $\mathbf{q}_3$ of part (a) when

$$\mathbf{q}_1 = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \mathbf{q}_2 = \frac{1}{2}\begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}, \mathbf{a}_3 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}.$$

$$\mathbf{q}_3 = \frac{1}{2}\begin{bmatrix} -1 \\ -1 \\ 1 \\ 1 \end{bmatrix}$$

10 pts

7. This problem uses least squares to find the line y = ax + b that best fits these 4 points in the plane:

$(x_1, y_1) = (-2,1)$, $(x_2, y_2) = (0,0)$, $(x_3, y_3) = (1,2)$, $(x_4, y_4) = (1,4)$.

a. Write down 4 equations $ax_i + b = y_i$, i = 1, 2, 3, 4, that would be true if the line actually went through all four points.

$$\begin{aligned} -2a + b &= 1 \\ 0a + b &= 0 \\ a + b &= 2 \\ a + b &= 4 \end{aligned}$$

8 pts

b. Now write those four equations in the form $\mathbf{A}\begin{bmatrix} a \\ b \end{bmatrix} = \mathbf{y}$

$$\boxed{\begin{bmatrix} -2 & 1 \\ 0 & 1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 2 \\ 4 \end{bmatrix}}$$

4 pts

c. Now find $\begin{bmatrix} \hat{a} \\ \hat{b} \end{bmatrix}$ that minimizes $\left\| \mathbf{A}\begin{bmatrix} a \\ b \end{bmatrix} - \mathbf{y} \right\|^2$.

$$\boxed{\hat{a} = \frac{2}{3} \quad \hat{b} = \frac{7}{4}}$$

14 pts

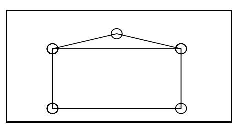
8. Let $f(x) = \det\begin{bmatrix} 5 & 0 & 3 \\ 2+x & 3x+1 & 4x-2 \\ 0 & 2 & 1 \end{bmatrix}$. Find f'(x). 12 pts

f'(x) = -19

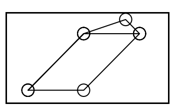
9. For each 2x2 matrix **A** below, draw a picture in the xy plane that shows A*house, where "house" is the set of points: $\{(0, 0), (2, 0), (2, 2), (0, 2), (1, 2.5)\}$ including lines for the floor, walls, ceiling, and roof, as shown below.

20 pts

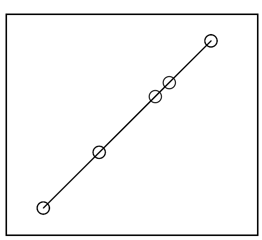
a. $A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$



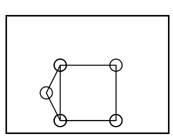
b. $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$



c. $A = \begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix}$



d. $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$



10. Find the complete solution to the system:

$$B = \begin{bmatrix} 1 & 2 & -1 & -5 \\ 2 & 4 & -1 & -8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \end{bmatrix} \qquad \boxed{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ -2 \\ 1 \end{bmatrix} + c_1 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 3 \\ 0 \\ -2 \\ 1 \end{bmatrix}}$$

12 pts

11. Find a subset of these vectors that forms a basis for the span of the vectors. Express the vectors not in the basis as combinations of the basis.

$$\mathbf{a}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \mathbf{a}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \\ -1 \\ -1 \end{bmatrix}, \mathbf{a}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \mathbf{a}_4 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}, \mathbf{a}_5 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ 1 \\ 3 \\ 1 \end{bmatrix} \qquad \boxed{\text{Basis } \mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_4}$$

$\boxed{\mathbf{a}_3 = (\mathbf{a}_1 + \mathbf{a}_2)/2, \quad \mathbf{a}_5 = \mathbf{a}_1 - \mathbf{a}_2 + \mathbf{a}_3}$. 20 pts

12. A 3x2 matrix (3 rows, 2 columns) A has a null space spanned by $\begin{bmatrix} 3 & 4 \end{bmatrix}$. The column space is spanned by $\begin{bmatrix} 2 & 1 & 2 \end{bmatrix}$. Also

$$A \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 4 \end{bmatrix}$$

a. Find a basis for the row space. $\boxed{\begin{bmatrix} -4 & 3 \end{bmatrix}}$ 8 pts

b. Use the SVD to find A. $\boxed{A = \frac{1}{7} \begin{bmatrix} 16 & -12 \\ 8 & -6 \\ 16 & -12 \end{bmatrix}}$ 10 pts

13. Find the eigenvalues and one real eigenvector of this permutation matrix.

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \qquad \boxed{\lambda = 1, \frac{-1}{2} \pm \frac{\sqrt{3}}{2} i, \quad v = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \text{ is an eigenvector}}$$

15 pts

14. This symmetric Markov matrix has zero determinant:

$$A = \begin{bmatrix} .4 & .2 & .4 \\ .2 & .6 & .2 \\ .4 & .2 & .4 \end{bmatrix}$$

a. What are the eigenvalues of A? $\boxed{\lambda = 1, 0.4, 0}$ 10 pts

b. Find $\lim_{k\to\infty} \mathbf{A}^k \mathbf{u_0}$ with $u_0 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$ $\boxed{\frac{1}{3}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}$ 15 pts

15. $A = \begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix}$

a. Find all of the eigenvalues of A. $\lambda = 4, 2$ 6 pts

b. Find a complete set of unit eigenvectors of A. 8 pts

$\lambda = 4: \ \frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ $\lambda = 2: \ \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

c. Find orthogonal matrices **U** and **V**, and diagonal Σ so that

$\mathbf{A} = \mathbf{U\Sigma V}^T$ $\boxed{\mathbf{U} = \mathbf{V} = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix},\ \Sigma = \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix}}$