SYMMETRIC HYPERBOLIC EQUATIONS OF MOTION FOR A HYPERELASTIC MATERIAL

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Abstract. We offer an alternate derivation for the symmetric-hyperbolic formulation of the equations of motion for a hyperelastic material with polyconvex stored energy. The derivation makes it clear that the expanded system is equivalent, for weak solutions, to the original system. We consider motions with variable as well as constant temperature. In addition, we present equivalent Eulerian equations of motion, which are also symmetric-hyperbolic.

1. Introduction: Kinematics

In [21] we demonstrated that the deformation gradient, which plays an important role in continuum mechanics, satisfies 22 divergence form conservation laws in the sense of distributions. These equations are consistent. In fact, all of these equations are implied, even for weak, L^∞ solutions (Lipschitz motions), by the continuity equations for a deformation gradient.

In 1990, Lefloch proposed to append additional conservation laws to the equations of elastodynamics (see [4] p. 121 and [17] p. 252). In [17], Qin showed that the 19 evolution equations for the deformation gradient, its cofactor matrix, and its determinant, together with the standard equations for conservation of momentum, form a symmetric-hyperbolic system when the material is hyperelastic with a polyconvex stored energy. In [7], Demoulini, Stuart, and Tzavaras used a variational approximation scheme to prove the existence of global measure-valued solutions to the symmetrized system of [17].

In this paper, we present an alternate derivation of this symmetric-hyperbolic system, and we extend it to motions with variable temperature. We prove that the symmetric-hyperbolic extension is equivalent to the original system for weak, L^∞ solutions. We present Eulerian as well as Lagrangian formulations of the extended system.

The work presented here is closely related to the theory of *involutions* [5, 4], and we shall try to elucidate this relationship as we proceed.

We begin with a brief introduction. Let $(\mathbf{a}, t)$ be coordinates for a reference configuration, and let $\mathbf{x} = \boldsymbol{\phi}(\mathbf{a}, t)$ be a motion—that is, $\mathbf{x} = \boldsymbol{\phi}(\mathbf{a}, t)$ describes the location in $\mathbb{R}^3$, at time t, of the material point labelled by $\mathbf{a}$. We suppose that the mapping

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$\boldsymbol{\phi}$ is locally Lipschitz continuous and one-to-one, with a locally Lipschitz inverse (*bi-Lipschitz*). Then, by Rademacher's theorem [8], $\boldsymbol{\phi}$ is differentiable almost everywhere. $\mathbf{F} = \nabla\boldsymbol{\phi} = \dfrac{\partial \mathbf{x}}{\partial \mathbf{a}}$ is a 3×3 matrix called the *deformation gradient*, and $\mathbf{v} = \dfrac{\partial \mathbf{x}}{\partial t}$ is the velocity.

We denote the i^{th} row of $\mathbf{F}$ by $\mathbf{F}^i$, and we denote the α^{th} column of $\mathbf{F}$ by $\mathbf{F}_\alpha$. The (i,α) entry of $\mathbf{F}$ we denote by F^i_α.

Then equality of mixed partial derivatives implies that $\dfrac{\partial}{\partial t} F^i_\alpha = \dfrac{\partial}{\partial a_\alpha} v^i$, or:

$$\frac{\partial \mathbf{F}}{\partial t} = \nabla_{\mathbf{a}} \otimes \mathbf{v}. \tag{1.1}$$

We will call this the continuity equation for $\mathbf{F}$. As noted in [5] and also shown in [16], if $\mathbf{F}$ satisfies (1.1) for $t > 0$, and at $t = 0$,

$$\frac{\partial F^i_\alpha}{\partial a^\beta} = \frac{\partial F^i_\beta}{\partial a^\alpha}, \tag{1.2}$$

then (1.2) holds for all $t > 0$. In [5], (1.2) is called an *involution* of (1.1). Since (1.1) and (1.2) are linear constant coefficient equations, this conclusion holds in the sense of distributions for any initial data $\mathbf{F}(\mathbf{a}, 0)$ in $\mathcal{D}'(\mathbb{R}^3)$.

We note that (1.1, 1.2) imply that

$$\omega^i = F^i_\alpha da^\alpha + v^i dt \tag{1.3}$$

is a closed differential one form. In fact, $\mathbf{F} = \dfrac{\partial \mathbf{x}}{\partial \mathbf{a}}$ and $\mathbf{v} = \dfrac{\partial \mathbf{x}}{\partial t}$ imply that ω^i is the exact form dx^i. In [21] we proved the following theorem:

Theorem 1.1. *Let $\alpha \in L^p_{loc}$ and $\beta \in L^{p'}_{loc}$ be differential forms of degree q and q', respectively, where $\dfrac{1}{p} + \dfrac{1}{p'} = 1$, $1 \le p \le \infty$ and further suppose that $d\alpha \in L^p_{loc}$ and $d\beta \in L^{p'}_{loc}$. Then $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^q \alpha \wedge d\beta$, and is locally integrable.*

The proof given for this theorem in [21] is valid for $1 < p < \infty$. Here we give a proof that is also valid for the case $p = 1$, $p' = \infty$.

Proof. Choose a coordinate patch, and within this patch, let $K \subset \mathbb{R}^n$ be an open set with compact closure. Standard localization and smoothing gives a sequence of smooth differential forms α_n, β_n such that on K, $\alpha_n \to \alpha$ and $d\alpha_n \to d\alpha$ in $L^1(K)$, while $\beta_n \to \beta$ and $d\beta_n \to d\beta$ in $L^r(K)$, $1 \le r < \infty$, and weak* in $L^\infty(K)$. Furthermore each sequence converges pointwise to its limit on the Lebesgue set of the limit function. We require the following lemma.

Lemma 1.2. *On K, $\alpha_n \wedge \beta_n \to \alpha \wedge \beta$, $d\alpha_n \wedge \beta_n \to d\alpha \wedge \beta$, and $\alpha_n \wedge d\beta_n \to \alpha \wedge d\beta$ in L^1.*

Proof. (Lemma) We prove the case for $\alpha_n \wedge \beta_n$. Note that:

$$\alpha_n \wedge \beta_n - \alpha \wedge \beta = (\alpha_n - \alpha) \wedge \beta_n + \alpha \wedge (\beta_n - \beta).$$

Looking at the convergence of the first term, we have

$$\|(\alpha_n - \alpha) \wedge \beta_n\|_1 \leq \|\alpha_n - \alpha\|_1 \|\beta_n\|_\infty \to 0, \tag{1.4}$$

since β_n is bounded in $L^\infty(K)$. For the second term, we have that pointwise,

$$\|\alpha \wedge (\beta_n - \beta)\|(x) \leq \max_n \|\beta_n - \beta\|_\infty \|\alpha(x)\| \in L^1(K).$$

Since $\beta_n \to \beta$ on the Lebesgue set of β (that is, almost everywhere), we have by the Lebesgue dominated convergence theorem that $\alpha \wedge (\beta_n - \beta) \to 0$ in $L^1(K)$. □

As a consequence of Lemma 1.2, we have that as distributions, $d(\alpha \wedge \beta) = \lim_{n\to\infty} d(\alpha_n \wedge \beta_n)$. Moreover,

$$d(\alpha_n \wedge \beta_n) = d\alpha_n \wedge \beta_n + (-1)^q \alpha_n \wedge d\beta_n. \tag{1.5}$$

By Lemma 1.2, the right side of (1.5) converges in $L^1(K)$ and hence also as distributions, to $d\alpha \wedge \beta + (-1)^q \alpha \wedge d\beta$. □

We define the *cofactor matrix* for $\mathbf{F}$, to be $(\mathbf{Cof\,F})^i_\alpha = (-1)^{i+\alpha} \det\left({\mathbf{F}^i_\alpha}'\right)$, where ${\mathbf{F}^i_\alpha}'$ is the 2×2 matrix obtained from $\mathbf{F}$ by deleting the i^{th} row and α^{th} column from $\mathbf{F}$. When $\mathbf{F}$ is invertible, as we have assumed, then $\mathbf{Cof\,F} = |\mathbf{F}|\,\mathbf{F}^{-T}$ [2].

We also define the *Lagrangian velocity*,

$$\mathbf{u} = \frac{\partial}{\partial t}\mathbf{a}(\mathbf{x}, t). \tag{1.6}$$

Since $0 = \frac{\partial}{\partial t}\mathbf{x}\left(\mathbf{a}(\mathbf{x}, t), t\right) = \frac{\partial \mathbf{x}}{\partial \mathbf{a}}\frac{\partial \mathbf{a}}{\partial t} + \frac{\partial}{\partial t}\mathbf{x}(\mathbf{a}, t)$,

$$\begin{aligned}\mathbf{u} &= -\mathbf{F}^{-1}\mathbf{v},\\ \mathbf{v} &= -\mathbf{F}\mathbf{u}.\end{aligned} \tag{1.7}$$

We are particularly interested in the case where $\mathbf{v},\ \mathbf{F} \in L^\infty_{loc}$ and satisfy (1.1, 1.2). In this case, $dx^i \in L^\infty_{loc}$ and is closed. Theorem 1.1, together with (1.3), implies that the following differential forms are also closed:

(1.8)

$$dx^i \wedge dx^j = \sum_{1\leq\alpha<\beta\leq 3} \left(F^i_\alpha F^j_\beta - F^i_\beta F^j_\alpha\right) da^\alpha \wedge da^\beta + \sum_{\alpha=1}^{3} \left(F^i_\alpha v^j - F^j_\alpha v^i\right) da^\alpha \wedge dt$$

$$\begin{aligned} dx^1 \wedge dx^2 \wedge dx^3 &= |\mathbf{F}|\, da^1 \wedge da^2 \wedge da^3 - \sum_{\alpha=1}^{3}\left(\sum_{i=1}^{3} v^i \left(\mathbf{Cof\,F}\right)^i_\alpha (-1)^\alpha\right) dt \wedge da_{\hat{\alpha}} \\ &= |\mathbf{F}|\, da^1 \wedge da^2 \wedge da^3 + u^\alpha\, |\mathbf{F}|\, dt \wedge da_{\hat{\alpha}} (-1)^\alpha \end{aligned}$$

Here $da_{\hat{1}} = da_2 \wedge da_3$, $da_{\hat{2}} = da_1 \wedge da_3$, and $da_{\hat{3}} = da_1 \wedge da_2$.

This implies that the following divergence-form differential equations hold, in the sense of distributions, for any deformation gradient $\mathbf{F} \in L^\infty_{loc}$ and $\mathbf{v}$ (or $\mathbf{u}$) $\in L^\infty_{loc}$ satisfying (1.1, 1.2):

$$\frac{\partial}{\partial t}\left(F^i_\alpha F^j_\beta - F^i_\beta F^j_\alpha\right) + \frac{\partial}{\partial a^\alpha}\left(F^i_\beta v^j - F^j_\beta v^i\right) - \frac{\partial}{\partial a^\beta}\left(F^i_\alpha v^j - F^j_\alpha v^i\right) = 0, \quad 1 \le i < j \le 3, \qquad 1 \le \alpha < \beta \le 3, \tag{1.9}$$

$$\frac{\partial}{\partial a^1}\left(F^i_2 F^j_3 - F^i_3 F^j_2\right) - \frac{\partial}{\partial a^2}\left(F^i_1 F^j_3 - F^i_3 F^j_1\right) + \frac{\partial}{\partial a^3}\left(F^i_1 F^j_2 - F^i_2 F^j_1\right) = 0, \quad 1 \le i < j \le 3, \tag{1.10}$$

$$\frac{\partial}{\partial t}|\mathbf{F}| + \sum_{\alpha=1}^{3} \frac{\partial}{\partial a^\alpha}\left(\mathbf{u}^\alpha |\mathbf{F}|\right) = 0 \tag{1.11}$$

Equation (1.10) is the Piola identity $\nabla \cdot \mathbf{Cof\,F} = 0$. [2, 15, 21].

Equation (1.11) is a conservation law for $|\mathbf{F}|$ which describes the conservation of volume in Lagrangian coordinates.

Equation (1.9) is a conservation law for $\mathbf{Cof\,F}$. We note that the Piola law (1.10) is an involution for (1.9). The Piola law appears to be a new involution, but because it expresses the fact that $dx_i \wedge dx_j$ is closed as a form on $\mathbb{R}^3$, it is implied in the sense of distributions by the involutions (1.2), together with the conservation law (1.1).

We note that Dafermos ([4], p. 30-31) and Demoulini, Stuart, and Tzavaras ([7]) proved that (1.9, 1.10, 1.11) hold in the sense of distributions. However we feel that the use of differential forms provides additional understanding of the matter.

Using (1.7), equation (1.9) can be rewritten as:

$$\begin{aligned} \frac{\partial}{\partial t}\left(F^i_\alpha F^j_\beta - F^i_\beta F^j_\alpha\right) &- \frac{\partial}{\partial a^\alpha}\left(\left(F^i_\beta F^j_\gamma - F^j_\beta F^i_\gamma\right) u^\gamma\right) \\ &+ \frac{\partial}{\partial a^\beta}\left(\left(F^i_\alpha F^j_\gamma - F^j_\alpha F^i_\gamma\right) u^\gamma\right) = 0, \\ 1 \le i < j \le 3, &\qquad 1 \le \alpha < \beta \le 3, \end{aligned} \tag{1.12}$$

where γ is summed from 1 to 3.

When i and j are distinct integers from 1 to 3, let $n(i,j) = 6 - i - j$. Then $n(i,j)$ is the integer from 1 to 3 which is neither i nor j.

We wish to find a simple expression for (1.12). Let

$$A^{ij}_\alpha = \left(F^i_\alpha v^j - F^j_\alpha v^i\right) = -\sum_{\gamma=1}^{3}\left(F^i_\alpha F^j_\gamma - F^j_\alpha F^i_\gamma\right) u^\gamma. \tag{1.13}$$

Equation (1.12) can be written:

$$\begin{aligned}
&\frac{\partial}{\partial t}\left(\mathbf{Cof\,F}^{n(i,j)}_{n(\alpha,\beta)}\right)(-1)^{n(i,j)+n(\alpha,\beta)} + \frac{\partial}{\partial a^\alpha}\left(A^{ij}_\beta\right) - \frac{\partial}{\partial a^\beta}\left(A^{ij}_\alpha\right) = 0, \text{ or} \\
&\frac{\partial}{\partial t}\left(\mathbf{Cof\,F}^{n(i,j)}_{n(\alpha,\beta)}\right)(-1)^{n(i,j)+n(\alpha,\beta)} - \left(\nabla_{\mathbf{a}} \times \mathbf{A}^{ij}\right)_{n(\alpha,\beta)}(-1)^{n(\alpha,\beta)} = 0, \\
&\qquad\qquad 1 \le i < j \le 3, \qquad 1 \le \alpha < \beta \le 3.
\end{aligned} \tag{1.14}$$

Since, for each fixed α,

$$A^{ij}_\alpha = -\left(\mathbf{F}_\alpha \times \mathbf{v}\right)^{n(i,j)}(-1)^{n(i,j)}, \tag{1.15}$$

equation (1.12) has the simple form,

$$\frac{\partial}{\partial t}\left(\mathbf{Cof\,F}\right) + \nabla_{\mathbf{a}} \times \left(\mathbf{F} \times \mathbf{v}\right) = 0, \tag{1.16}$$

where $\mathbf{F} \times \mathbf{v}$ denotes the 3×3 matrix $\mathbf{B}$ for which the column vector $\mathbf{B}_\alpha = \mathbf{F}_\alpha \times \mathbf{v}$, and $\nabla_{\mathbf{a}} \times \mathbf{B}$ is the matrix $\mathbf{C}$ for which the row vector $\mathbf{C}^i = \nabla_{\mathbf{a}} \times \mathbf{B}^i$.

We can also express A^{ij}_α in terms of $\mathbf{Cof\,F}$:

$$\begin{aligned}
A^{ij}_\alpha (-1)^{n(i,j)} &= \left(\mathbf{F}_\alpha \times \mathbf{F}\mathbf{u}\right)^{n(i,j)}, \\
&= \sum_{\gamma=1}^{3}\left(\mathbf{F}_\alpha \times \mathbf{F}_\gamma u^\gamma\right)^{n(i,j)}, \\
&= \sum_{\gamma\neq\alpha}\left(\mathbf{F}_\alpha \times \mathbf{F}_\gamma\right)^{n(i,j)}_{n(\alpha,\gamma)} u^\gamma, \\
&= -\sum_{1\le\gamma<\alpha}\left(\mathbf{Cof\,F}\right)^{n(i,j)}_{n(\alpha,\gamma)} u^\gamma (-1)^{n(\alpha,\gamma)} + \sum_{\alpha<\gamma\le 3}\left(\mathbf{Cof\,F}\right)^{n(i,j)}_{n(\alpha,\gamma)} u^\gamma (-1)^{n(\alpha,\gamma)}, \\
&= \left(\mathbf{Cof\,F}^{n(i,j)} \times \mathbf{u}\right)_\alpha.
\end{aligned} \tag{1.17}$$

Thus, equation (1.12) can be written simply as

$$\begin{aligned}
&\frac{\partial}{\partial t}\left(\mathbf{Cof\,F}\right) + \nabla_{\mathbf{a}} \times \left(\mathbf{F} \times \mathbf{v}\right) = 0, \text{ or} \\
&\frac{\partial}{\partial t}\left(\mathbf{Cof\,F}\right) - \nabla_{\mathbf{a}} \times \left(\mathbf{Cof\,F} \times \mathbf{u}\right) = 0.
\end{aligned} \tag{1.18}$$

Note, however, that in the second equation of (1.18), the cross product $\mathbf{Cof\,F} \times \mathbf{u}$ is a matrix $\mathbf{B}$ for which the row vector $\mathbf{B}^{n(i,j)} = \left(\mathbf{Cof\,F}\right)^{n(i,j)} \times \mathbf{u}$. Also note that this equation bears a superficial similarity with the vorticity equation of incompressible Euler flow, $\omega_t + \nabla \times (\omega \times \mathbf{v}) = 0$—the difference being the minus sign, and the relationship $\omega = \nabla \times \mathbf{v}$, which is not present in the cofactor equation.

2. Elasticity and Hyperelasticity

A material is called *elastic* if the first Piola-Kirchhoff stress tensor has the form [2, 15, 19]

$$T(\mathbf{a}) = \hat{T}\big(\mathbf{a}, \mathbf{F}(\mathbf{a})\big) \tag{2.1}$$

for all $\mathbf{a}$. An elastic material is called *hyperelastic* if, in addition to (2.1), the first Piola-Kirchhoff stress has the form

$$\hat{T}\big(\mathbf{a}, \mathbf{F}\big) = \frac{\partial \hat{W}}{\partial \mathbf{F}}\big(\mathbf{a}, \mathbf{F}\big). \tag{2.2}$$

The function $\hat{W}$ is called the *stored energy function.* Natural physical assumptions imply that $\hat{W}$ cannot be convex; see [3] and [19]. However, it was discovered by John Ball [1] that it is physically consistent to assume that $\hat{W}$ is *polyconvex*; that is, $\hat{W}$ has the form:

$$\hat{W}(\mathbf{a}, \mathbf{F}) = \mathbb{W}(\mathbf{a}, \mathbf{F}, \mathbf{Cof}\,\mathbf{F}, |\mathbf{F}|), \tag{2.3}$$

where $\mathbb{W}$ is convex with respect to $\mathbf{F}, \mathbf{Cof}\,\mathbf{F}, |\mathbf{F}|$.

The *equations of motion* for an isothermal elastic material are:

$$\frac{\partial^2 \mathbf{x}}{\partial t^2} = \nabla_{\mathbf{a}} \cdot \hat{T}\Big(\mathbf{a}, \frac{\partial \mathbf{x}}{\partial \mathbf{a}}(\mathbf{a}, t)\Big) \tag{2.4}$$

This system is discussed in [12, 6]. If the operator $x \to \nabla_{\mathbf{a}} \cdot \hat{T}\Big(\mathbf{a}, \frac{\partial \mathbf{x}}{\partial \mathbf{a}}(\mathbf{a}, t)\Big)$ is strongly elliptic, (2.4) is a hyperbolic system. For a hyperelastic system, the operator is strongly elliptic if and only if $\hat{W}$ is *rank one convex* [5, 4]. Using (1.1), we can write this as a first order system, as follows:

$$\begin{aligned} \frac{\partial \mathbf{F}}{\partial t} &= \nabla_{\mathbf{a}} \otimes \mathbf{v} \\ \frac{\partial \mathbf{v}}{\partial t} &= \nabla_{\mathbf{a}} \cdot \hat{T}\big(\mathbf{a}, \mathbf{F}(\mathbf{a}, t)\big) \end{aligned} \tag{2.5}$$

Qin [17] showed that if the material is hyperelastic with a polyconvex stored energy function, then we can enlarge (2.5), by adding equation (1.11) and the nine equations (1.9), to obtain a system which has a convex extension in the sense of Lax and Friedrichs [9, 13]. Theorem 2.4 shows that Qin's enlarged system is equivalent to (2.5) for weak solutions.

Using the velocity $\mathbf{v}$, the enlarged system is:

$$\begin{aligned}
&\frac{\partial \mathbf{F}}{\partial t} - \nabla_{\mathbf{a}} \otimes \mathbf{v} = 0, \\
&\frac{\partial \mathbf{Cof\,F}}{\partial t} + \nabla_{\mathbf{a}} \times \big(\mathbf{F} \times \mathbf{v}\big) = 0, \\
&\frac{\partial \left|\mathbf{F}\right|}{\partial t} - \nabla_{\mathbf{a}} \cdot \big(\mathbf{F}^{-1}\mathbf{v}\left|\mathbf{F}\right|\big) = 0, \\
&\frac{\partial \mathbf{v}}{\partial t} - \nabla_{\mathbf{a}} \cdot \frac{\partial \hat{W}}{\partial \mathbf{F}}\big(\mathbf{a}, \mathbf{F}(\mathbf{a}, t)\big) = 0.
\end{aligned} \tag{2.6}$$

This is the same as the enlarged system derived in [17], and studied in [7].

In terms of the Lagrangian velocity $\mathbf{u}$, the system is:

$$\begin{aligned}
&\frac{\partial \mathbf{F}}{\partial t} + \nabla_{\mathbf{a}} \otimes \mathbf{F}\mathbf{u} = 0, \\
&\frac{\partial \mathbf{Cof\,F}}{\partial t} - \nabla_{\mathbf{a}} \times \big(\mathbf{Cof\,F} \times \mathbf{u}\big) = 0, \\
&\frac{\partial \left|\mathbf{F}\right|}{\partial t} + \nabla_{\mathbf{a}} \cdot \left(\mathbf{u}\left|\mathbf{F}\right|\right) = 0, \\
&\frac{\partial \mathbf{F}\mathbf{u}}{\partial t} + \nabla_{\mathbf{a}} \cdot \frac{\partial \hat{W}}{\partial \mathbf{F}}\big(\mathbf{a}, \mathbf{F}(\mathbf{a}, t)\big) = 0.
\end{aligned} \tag{2.7}$$

The convex extension to (2.6) is the conservation of total energy. The total energy is the kinetic energy plus the stored energy. The conservation law is:

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\left\|\mathbf{v}\right\|^2 + \hat{W}(\mathbf{a}, \mathbf{F})\right) - \nabla_{\mathbf{a}} \cdot \left(\mathbf{v} \cdot \frac{\partial \hat{W}}{\partial \mathbf{F}}\big(\mathbf{a}, \mathbf{F}(\mathbf{a}, t)\big)\right) = 0. \tag{2.8}$$

That this conservation law is a consequence of (2.6) for smooth solutions, follows from the following calculation:

$$\begin{aligned}
\frac{\partial}{\partial t}\left(\frac{1}{2}\left\|\mathbf{v}\right\|^2 + \hat{W}(\mathbf{a}, \mathbf{F})\right) &= \mathbf{v} \cdot \mathbf{v}_t + \frac{\partial \hat{W}}{\partial \mathbf{F}} : \frac{\partial \mathbf{F}}{\partial t}, \\
&= \mathbf{v} \cdot \left(\nabla_{\mathbf{a}} \cdot \frac{\partial \hat{W}}{\partial \mathbf{F}}\right) + \frac{\partial \hat{W}}{\partial \mathbf{F}} : \nabla_{\mathbf{a}} \otimes \mathbf{v} \\
&= v^i \frac{\partial}{\partial a^\alpha} \frac{\partial \hat{W}}{\partial F^i_\alpha} + \frac{\partial \hat{W}}{\partial F^i_\alpha} \frac{\partial v^i}{\partial a^\alpha}, \\
&= \frac{\partial}{\partial a^\alpha}\left(v^i \frac{\partial \hat{W}}{\partial F^i_\alpha}\right), \\
&= \nabla_{\mathbf{a}} \cdot \left(\mathbf{v} \cdot \frac{\partial \hat{W}}{\partial \mathbf{F}}\right).
\end{aligned} \tag{2.9}$$

Because the continuity equation implies the conservation equations for the cofactor matrix and the determinant, and because it does this at the level of weak solutions, we can regard a polyconvex stored energy function as a convex function of conserved quantities. For discontinuous solutions, (2.8) will not hold as an equality. However an associated "entropy inequality" can be used as an admissibility criterion to eliminate spurious solutions:

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\|\mathbf{v}\|^2 + \mathbb{W}(\mathbf{a}, \mathbf{F}, \mathbf{Cof}\,\mathbf{F}, |\mathbf{F}|)\right) - \nabla_{\mathbf{a}} \cdot \left(\mathbf{v} \cdot \frac{\partial \hat{W}}{\partial \mathbf{F}}(\mathbf{a}, \mathbf{F}(\mathbf{a}, t))\right) \leq 0. \tag{2.10}$$

Thus, except for our use of redundant, but consistent, equations for the deformation gradient, the equations of motion for a hyperelastic material with polyconvex stored energy function fit into the standard framework for systems of hyperbolic conservation laws modeling physical phenomena.

We summarize these remarks in the following theorem.

Theorem 2.1. *If $\hat{W}(\mathbf{a}, \mathbf{F})$ is a polyconvex stored energy function, then the systems (2.6, 2.7) have a convex extension given by the conservation of total energy.*

One important consequence of the existence of a convex extension, is that Lax and Friedrichs [9] (see also Godunov [10], and Mock [11]) showed that such a system has a quasilinear form which is symmetric-hyperbolic. If the system has the form

$$\frac{\partial U}{\partial t} + \sum_{i=1}^{n} \frac{\partial F^i(U)}{\partial x^i} = 0, \tag{2.11}$$

with entropy inequality

$$\frac{\partial \eta(U)}{\partial t} + \sum_{i=1}^{n} \frac{\partial q^i(U)}{\partial x^i} \leq 0, \tag{2.12}$$

with $\eta(U)$ strictly convex, and $\nabla\eta(U) = \nabla q^i(U) \cdot DF^i(U)$, then multiplying (2.11) by $D^2\eta(U)$ yields the following system, valid for classical solutions:

$$D^2\eta(U)\frac{\partial U}{\partial t} + \sum_{i=1}^{n} D^2\eta(U) DF^i(U) \frac{\partial U}{\partial x^i} = 0, \tag{2.13}$$

where $D^2\eta(U)$ is symmetric positive-definite and $D^2\eta(U)DF^i(U)$ are all symmetric. For symmetric hyperbolic systems, the Cauchy problem is known to be well-posed for small smooth initial data, for short time [14]. Also, we are guaranteed that a full set of characteristic eigenvectors exist in every wave direction. Thus, for each non-zero vector $\mathbf{n} \in \mathbb{R}^n$, the matrix $DF^i n^i$ is diagonalizable.

Applying Theorem 2.1 of [14], we have the following Corollary to Theorem 2.1, which ignores the important issue of boundary conditions.

Corollary 2.2. *Let* $\mathbf{x} = \boldsymbol{\phi}_0(\mathbf{a})$ *be given, with* $\boldsymbol{\phi}_0$ *bi-Lipschitz on* $\mathbb{R}^3$*, and* $\boldsymbol{\phi}_0(\mathbf{a}) = \mathbf{a}$ *outside of a compact set. Let* $\mathbf{F}_0,\ \mathbf{Cof}\,\mathbf{F}_0$*, and* D_0 *be the deformation gradient of* $\boldsymbol{\phi}_0$ *and its cofactor matrix and determinant, respectively. Let* $\mathbf{v}_0(\mathbf{a})$ *be the initial velocity; assume* $\mathbf{v}_0(\mathbf{a}) = 0$ *outside of a compact set. Let* $\hat{W}(\mathbf{a}, \mathbf{F})$ *be a smooth strictly polyconvex stored energy function with domain* $\mathbb{R}^3 \times K$*, where* K *is a convex subset of* $\mathbb{R}^{19} \cap \{D > 0\}$*.*

Let $s > \frac{5}{2}$*. If* $U_0 = \big(\mathbf{F}_0 - I,\ \mathbf{Cof}\,\mathbf{F}_0 - I,\ D_0 - 1,\ \mathbf{v}_0\big) \in H^s(\mathbb{R}^3, \mathbb{R}^{22})$ *and* $\big(\mathbf{F}_0,\ \mathbf{Cof}\,\mathbf{F}_0,\ D_0,\ \mathbf{v}_0\big)$ *has values in a compact subset* G *of* $K \times \mathbb{R}^3$*, then there exists* $T > 0$*, such that the equations* (2.6) *have a unique classical solution* $U(x,t) \in C^1(R^3 \times [0,T],\ G)$ *with initial data* U_0*. Furthermore,*

$$U \in C([0,T], H^s) \cap C^1\left([0,t], H^{s-1}\right),$$

and T *depends on* $\|U_0\|_s$ *and* G*.*

Corollary 2.2 is very similar to the theorem proved regarding system (2.4) in [12]. The paper [6] proves existence for initial-*boundary* value problems for (2.4), but with stronger regularity requirements on the initial data. However, both [12, 6] only require the stress to be strongly elliptic, whereas Corollary 2.2 requires, in addition, that the stored energy be convex in $\mathbf{F},\ \mathbf{Cof}\,\mathbf{F}$, and D.

Dafermos' theory of *involutions* [5, 4] provides an alternative approach to the well-posedness of the Cauchy problem. For systems that are hyperbolic, but lack which lack a strictly convex extension, involutions can provide additional constraints on oscillations that compensate for this lack. Specifically, one needs a C^3 entropy $\eta(U)$ for which $D^2\eta(U)$ is positive definite in the direction of the "involution cone." Under such conditions Dafermos proved an existence theorem similar to Corollary 2.2. Again, for elasticity this Dafermos' theorem is more general than Corollary 2.2 because it does not require a polyconvex stored energy.

One can also consider the equations of motion and of conservation of energy for a hyperelastic material in which thermal effects are of interest. For such a material, the natural assumption on the stored energy function is that $\hat{W}(\mathbf{a}, \mathbf{F}, S) = \mathbb{W}(\mathbf{a}, \mathbf{F}, \mathbf{Cof}\,\mathbf{F}, |F|, S)$ is convex as a function of $\mathbf{F}, \mathbf{Cof}\,\mathbf{F}, |F|$ and S, where S is the density of entropy with respect to mass [4]. In this case, one still has

$$\begin{aligned} \hat{T}(\mathbf{a}, \mathbf{F}) &= \frac{\partial \hat{W}}{\partial \mathbf{F}}, \\ &= \frac{\partial \mathbb{W}}{\partial \mathbf{F}} + \frac{\partial \mathbb{W}}{\partial \mathbf{Cof}\,\mathbf{F}} \frac{\partial \mathbf{Cof}\,\mathbf{F}}{\partial \mathbf{F}} + \frac{\partial \mathbb{W}}{\partial |F|} \frac{\partial |F|}{\partial \mathbf{F}}. \end{aligned} \tag{2.14}$$

The absolute temperature is $\theta = \dfrac{\partial \hat{W}}{\partial S}$. We assume that $\theta > 0$. If the heat conductivity is zero, then the equations of motion and of conservation of energy are:

$$\begin{gathered}
\frac{\partial \mathbf{F}}{\partial t} - \nabla_{\mathbf{a}} \otimes \mathbf{v} = 0, \\
\frac{\partial \mathbf{Cof\,F}}{\partial t} + \nabla_{\mathbf{a}} \times \big(\mathbf{F} \times \mathbf{v}\big) = 0, \\
\frac{\partial \left|\mathbf{F}\right|}{\partial t} - \nabla_{\mathbf{a}} \cdot \big(\mathbf{F}^{-1}\mathbf{v}\left|\mathbf{F}\right|\big) = 0, \\
\frac{\partial \mathbf{v}}{\partial t} - \nabla_{\mathbf{a}} \cdot \frac{\partial \hat{W}}{\partial \mathbf{F}}\big(\mathbf{a}, \mathbf{F}(\mathbf{a}, t), S(\mathbf{a}, t)\big) = 0. \\
\frac{\partial}{\partial t}\left(\frac{1}{2}\left\|\mathbf{v}\right\|^2 + \mathbb{W}(\mathbf{a}, \mathbf{F}, \mathbf{Cof\,F}, \left|\mathbf{F}\right|, S)\right) - \nabla_{\mathbf{a}} \cdot \left(\mathbf{v} \cdot \frac{\partial \hat{W}}{\partial \mathbf{F}}\big(\mathbf{a}, \mathbf{F}(\mathbf{a}, t), S\big)\right) = 0.
\end{gathered} \tag{2.15}$$

and the convex extension is given by the entropy function $-S$; the equation is simply $S_t = 0$, and the admissibility criterion is $S_t \geq 0$. We require the following theorem, which is based on an idea due to Andrew Majda; see [20].

Theorem 2.3. *Let* $\mathbb{W}(\mathbf{a}, \mathbf{F}, \mathbf{Cof\,F}, \left|\mathbf{F}\right|, S)$ *be convex in* $(\mathbf{F}, \mathbf{Cof\,F}, \left|\mathbf{F}\right|, S)$*, and let* $E = \left\|\mathbf{v}\right\|^2/2 + \mathbb{W}$*. Suppose also that* $\theta = \dfrac{\partial \hat{W}}{\partial S} > 0$*. Then there exists a function* $S(\mathbf{a}, \mathbf{F},\ \mathbf{Cof\,F},\ \left|\mathbf{F}\right|,\ \mathbf{v}, E)$ *such that* $-S$ *is convex in* $(\mathbf{F},\ \mathbf{Cof\,F},\ \left|\mathbf{F}\right|,\ \mathbf{v},\ E)$*, and*

$$E = \left\|\mathbf{v}\right\|^2/2 + \mathbb{W}\big(\mathbf{a}, \mathbf{F}, \mathbf{Cof\,F}, \left|\mathbf{F}\right|, S(\mathbf{a}, \mathbf{F},\ \mathbf{Cof\,F},\ \left|\mathbf{F}\right|,\ \mathbf{v}, E)\big). \tag{2.16}$$

Proof. For $\mathbf{a}, \mathbf{F},\ \mathbf{Cof\,F},\ \left|\mathbf{F}\right|,\ \mathbf{v}$ fixed, the relationship between E and S is one to one since $\dfrac{\partial \hat{W}}{\partial S} > 0$. Therefore $S(\mathbf{a}, \mathbf{F},\ \mathbf{Cof\,F},\ \left|\mathbf{F}\right|,\ \mathbf{v},\ E)$ is uniquely determined by (2.16). We note that E is a convex function of $\mathbf{F},\ \mathbf{Cof\,F},\ \left|\mathbf{F}\right|,\ S$, and $\mathbf{v}$, and that the graph of S as a function of $\mathbf{F},\ \mathbf{Cof\,F},\ \left|\mathbf{F}\right|,\ \mathbf{v}$, and E is the same surface as the graph of E, with a simple exchange of the dependent variable E with the independent variable S. Thus S is either convex or concave. Since $\theta = \dfrac{\partial \hat{W}}{\partial S} > 0$, S must be concave and $-S$ is convex in $(\mathbf{F},\ \mathbf{Cof\,F},\ \left|\mathbf{F}\right|,\ \mathbf{v}, E)$. □

We may emphasize the independence of $\mathbf{F}$, $\mathbf{C} = \mathbf{Cof\,F}$, and $D = |\mathbf{F}|$ by rewriting (2.15) as:

$$
\begin{aligned}
\frac{\partial \mathbf{F}}{\partial t} - \nabla_{\mathbf{a}} \otimes \mathbf{v} = 0, \\
\frac{\partial \mathbf{C}}{\partial t} + \nabla_{\mathbf{a}} \times \big(\mathbf{F} \times \mathbf{v}\big) = 0, \\
\frac{\partial D}{\partial t} - \nabla_{\mathbf{a}} \cdot \big(\mathbf{F}^{-1}\mathbf{v} D\big) = 0, \\
\frac{\partial \mathbf{v}}{\partial t} - \nabla_{\mathbf{a}} \cdot \frac{\partial \hat{W}}{\partial \mathbf{F}}\big(\mathbf{a}, \mathbf{F}(\mathbf{a},t), S(\mathbf{a},t)\big) = 0. \\
\frac{\partial}{\partial t}\left(\frac{1}{2}\|\mathbf{v}\|^2 + \mathbb{W}(\mathbf{a}, \mathbf{F}, \mathbf{C}, D, S)\right) - \nabla_{\mathbf{a}} \cdot \left(\mathbf{v}\frac{\partial \hat{W}}{\partial \mathbf{F}}\big(\mathbf{a}, \mathbf{F}(\mathbf{a},t), S\big)\right) = 0.
\end{aligned}
\tag{2.17}
$$

Note that the equation for D may be replaced by:

$$
\frac{\partial D}{\partial t} - \nabla_{\mathbf{a}} \cdot \big(\mathbf{C}^T \mathbf{v}\big) = 0. \tag{2.18}
$$

Initial conditions for the Cauchy problem for (2.6, 2.7), or (2.15) must be posed in a consistent manner. These systems are all symmetric hyperbolic, and the Cauchy problem is well posed for short time, for initial data which is sufficiently smooth and small. However, we must require that the initial data for $\mathbf{C}$ and D actually be the cofactor matrix and determinant of the initial data for $\mathbf{F}$. In addition, as was noted before, $\mathbf{F}$ must be a gradient at $t = 0$. The continuity equation then ensures that $\mathbf{F}$ remains a gradient for $t > 0$.

We have already seen that the cofactor matrix and determinant of $\mathbf{F}$ satisfy the equations for $\mathbf{C}$ and D. The question arises, however, whether $\mathbf{C}(\mathbf{a},t) = \mathbf{Cof\,F}(\mathbf{a},t)$ and $D(\mathbf{a},t) = |\mathbf{F}|(\mathbf{a},t)$ for $t > 0$. This is a question of uniqueness of solutions. Uniqueness of weak solutions to nonlinear systems of hyperbolic conservation laws is a difficult subject with important unresolved issues. However, if we treat $\mathbf{F}$, $\mathbf{v}$ and S as known, then the conservation laws for $\mathbf{C}$ and D together with initial data, are linear in those unknowns, and determine $\mathbf{C}(\mathbf{a},t)$ and $D(\mathbf{a},t)$ uniquely. We have proved the following theorem.

Theorem 2.4. *Let Ω be an open set in $\mathbb{R}^3$ and let $T > 0$. Let*

$$
\big(\mathbf{F}(\mathbf{a},t),\ \mathbf{C}(\mathbf{a},t),\ D(\mathbf{a},t),\ \mathbf{v}(\mathbf{a},t),\ S(\mathbf{a},t)\big)
$$

be an L^∞ solution of (2.17) on $\Omega \times [0,T]$, such that $\mathbf{C}(\mathbf{a},0) = \mathbf{Cof\,F}(\mathbf{a},0)$ and $D(\mathbf{a},0) = |\mathbf{F}|(\mathbf{a},0)$ for all $\mathbf{a} \in \Omega$. Then $\mathbf{C}(\mathbf{a},t) = \mathbf{Cof\,F}(\mathbf{a},t)$ and $D(\mathbf{a},t) = |\mathbf{F}|(\mathbf{a},t)$ in $\Omega \times [0,T]$.

In addition, if if there is a map $\mathbf{x} : \Omega \to \mathbb{R}^3$ such that, in the sense of distributions, $\mathbf{F}(\mathbf{a},0) = \nabla\mathbf{x}(\mathbf{a})$, then $\mathbf{x}$ extends to a Lipschitz map on $\Omega \times [0,T]$ such that $\mathbf{F}(\mathbf{a},t) = \nabla\mathbf{x}(\mathbf{a},t)$ and $\mathbf{v}(\mathbf{a},t) = \dfrac{\partial \mathbf{a}}{\partial t}$.

3. Eulerian equations

The papers [16, 18] discussed the Eulerian formulation of the equations of motion for elastic materials. Our paper [21] clarified an incompatibility between these two papers regarding Eulerian continuity equations. Here, we briefly outline the derivation of the Eulerian equations, using differential forms, and we add the results of Section 2 regarding convex extensions.

In Eulerian coordinates $(\mathbf{x}, t)$, the forms $dx_i, dx_i \wedge dx_j$, and $dx_1 \wedge dx_2 \wedge dx_3$ are closed and exact, trivially. There is nothing to be gained by transforming the Lagrangian equations corresponding to these differential forms into Eulerian coordinates.

Significant partial differential equations can be derived, however, from the forms da_α, $da_\alpha \wedge da_\beta$, and $da_1 \wedge da_2 \wedge da_3$. These equations are the same as (1.9, 1.10, 1.11), with $\mathbf{F}$ replaced by $\mathbf{F}^{-1}$, $\mathbf{a}$ by $\mathbf{x}$, and $\mathbf{v}$ by $\mathbf{u}$ or by $-\mathbf{F}^{-1}\mathbf{v}$—the algebra is exactly the same. Let $\mathbf{G} = \mathbf{F}^{-1}$. The continuity equations come from da_α:

$$\frac{\partial}{\partial t} G_i^\alpha = \frac{\partial u^\alpha}{\partial x^i}, \tag{3.1}$$

with involution

$$\frac{\partial}{\partial x^i} G_j^\alpha = \frac{\partial}{\partial x^j} G_i^\alpha \tag{3.2}$$

The cofactor equations come from $da_\alpha \wedge da_\beta$:

$$\frac{\partial}{\partial t}\left(G_i^\alpha G_j^\beta - G_j^\alpha G_i^\beta\right) + \frac{\partial}{\partial x^i}\left(G_j^\alpha u^\beta - G_j^\beta u^\alpha\right) - \frac{\partial}{\partial x^j}\left(G_i^\alpha u^\beta - G_i^\beta u^\alpha\right) = 0, \quad 1 \le i < j \le 3, \qquad 1 \le \alpha < \beta \le 3, \tag{3.3}$$

with involution

$$\frac{\partial}{\partial x^1}\left(G_2^i G_3^j - G_3^i G_2^j\right) - \frac{\partial}{\partial x^2}\left(G_1^i G_3^j - G_3^i G_1^j\right) + \frac{\partial}{\partial x^3}\left(G_1^i G_2^j - G_2^i G_1^j\right) = 0, \tag{3.4}$$

The conservation of mass comes from $da_1 \wedge da_2 \wedge da_3$:

$$\frac{\partial}{\partial t}|\mathbf{G}| + \sum_{i=1}^{3} \frac{\partial}{\partial x^i}\left(\mathbf{v}^i |\mathbf{G}|\right) = 0 \tag{3.5}$$

In fact we can proceed directly to the following forms for these equations. Using the Lagrangian velocity $\mathbf{u}$, the equations for $\mathbf{G}$, $\mathbf{Cof\,G}$, and $|\mathbf{G}|$ are:

$$\begin{aligned} &\frac{\partial \mathbf{G}}{\partial t} - \nabla_{\mathbf{x}} \otimes \mathbf{u} = 0, \\ &\frac{\partial \mathbf{Cof\,G}}{\partial t} + \nabla_{\mathbf{x}} \times \left(\mathbf{G} \times \mathbf{u}\right) = 0, \\ &\frac{\partial |\mathbf{G}|}{\partial t} - \nabla_{\mathbf{x}} \cdot \left(\mathbf{G}^{-1}\mathbf{u}\,|\mathbf{G}|\right) = 0, \end{aligned} \tag{3.6}$$

In terms of the Eulerian (i.e., usual) velocity $\mathbf{v}$, these equations are:

$$\begin{aligned} &\frac{\partial \mathbf{G}}{\partial t} + \nabla_{\mathbf{x}} \otimes \mathbf{G}\mathbf{v} = 0, \\ &\frac{\partial \mathbf{Cof\, G}}{\partial t} - \nabla_{\mathbf{x}} \times \big(\mathbf{Cof\, G} \times \mathbf{v}\big) = 0, \\ &\frac{\partial \left|\mathbf{G}\right|}{\partial t} + \nabla_{\mathbf{x}} \cdot (\mathbf{v}\left|\mathbf{G}\right|) = 0, \end{aligned} \tag{3.7}$$

Equation (3.5) and the corresponding equations of (3.6, 3.7) describe the conservation of mass.

Thus, for kinematics, there is complete symmetry between the Lagrangian and Eulerian equations for continuity, for the cofactor matrix, and for the conservation of mass or volume. The conservation of momentum, however, is a physical law which is not tied to any coordinate system. Therefore we must take the differential form corresponding to this conservation law, and express it in the Eulerian coordinate system to obtain partial differential equations.

The conservation of momentum equation is (2.5):

$$\frac{\partial \mathbf{v}}{\partial t} = \nabla_{\mathbf{a}} \cdot \hat{T}\big(\mathbf{a}, \mathbf{F}(\mathbf{a}, t)\big). \tag{3.8}$$

The closed differential form corresponding to this equation is

$$\boldsymbol{\mu} = \mathbf{v} da_1 \wedge da_2 \wedge da_3 - \sum_{\alpha=1}^{3} \hat{\mathbf{T}}^{\alpha} dt \wedge da_{\hat{\alpha}} (-1)^{\alpha} \tag{3.9}$$

The conservation of mass (3.7) says that the following differential form, which we call the mass form, is closed:

$$\boldsymbol{\rho} = da_1 \wedge da_2 \wedge da_3 = \left|\mathbf{G}\right| dx_1 \wedge dx_2 \wedge dx_3 + \sum_{i=1}^{3} v^i \left|\mathbf{G}\right| dt \wedge dx_{\hat{i}} (-1)^i. \tag{3.10}$$

Then,

$$\begin{aligned} \boldsymbol{\mu} &= \mathbf{v}\boldsymbol{\rho} - \sum_{\alpha=1}^{3} \hat{\mathbf{T}}^{\alpha} dt \wedge da_{\hat{\alpha}} (-1)^{\alpha} \\ &= \mathbf{v} \left(\left|\mathbf{G}\right| dx_1 \wedge dx_2 \wedge dx_3 + \sum_{i=1}^{3} v^i \left|\mathbf{G}\right| dt \wedge dx_{\hat{i}} (-1)^i \right) - \sum_{\alpha=1}^{3} \hat{\mathbf{T}}^{\alpha} dt \wedge da_{\hat{\alpha}} (-1)^{\alpha} \end{aligned} \tag{3.11}$$

To transform the last term of (3.11), we either compute that

$$dt \wedge da_{\hat{\alpha}} (-1)^{\alpha} = dt \wedge \sum_{i=1}^{3} \mathbf{Cof\, G}_i^{\alpha} dx_{\hat{i}} (-1)^i,$$

or apply the Piola transform [2, 15, 19]: $\hat{\mathbf{T}}^{\phi}(x,t) = \hat{\mathbf{T}}(a,t)\mathbf{Cof\,G}$. We obtain that

$$\boldsymbol{\mu} = \left(\mathbf{v}\,|\mathbf{G}|\, dx_1 \wedge dx_2 \wedge dx_3 + \sum_{i=1}^{3} \Big(\mathbf{v} v^i\,|G| - \hat{\mathbf{T}}(a(x,t),t)\mathbf{Cof\,G} \Big)\, dt \wedge dx_{\hat{i}} (-1)^i \right) \tag{3.12}$$

Thus, the Eulerian equations of motion take the form:

$$\begin{aligned} \frac{\partial}{\partial t}\mathbf{G} + \nabla_{\mathbf{x}} \otimes (\mathbf{G}\mathbf{v}) &= 0, \\ \frac{\partial}{\partial t}\mathbf{Cof\,G} - \nabla_{\mathbf{x}} \times (\mathbf{Cof\,G} \times \mathbf{v}) &= 0, \\ \frac{\partial}{\partial t}\,|\mathbf{G}| + \nabla_{\mathbf{x}} \cdot (|\mathbf{G}|\,\mathbf{v}) &= 0, \\ \frac{\partial}{\partial t}\,(|\mathbf{G}|\,\mathbf{v}) + \nabla_{\mathbf{x}} \cdot \Big(|\mathbf{G}|\,\mathbf{v} \otimes \mathbf{v} - \hat{\mathbf{T}}\ \mathbf{Cof\,G} \Big) &= 0. \end{aligned} \tag{3.13}$$

Furthermore, when the material is hyperelastic with a polyconvex stored energy function $\mathbb{W}\big(\mathbf{F}, \mathbf{Cof\,F}, |\mathbf{F}|\big)$, the Eulerian energy density provides a convex extension, as follows. The energy density is $E^{\phi} = |G|\,\big(\|\mathbf{v}\|^2/2 + \mathbb{W}\big)$. The following theorem is a paraphrase of a result of [20].

Theorem 3.1. *Let f be convex on a subset Ω of $\mathbb{R}^+ \times \mathbb{R}^{n-1}$. Let $\boldsymbol{\phi} : \Omega \to \Omega^{\phi} \subset \mathbb{R}^+ \times \mathbb{R}^{n-1}$ be defined by*

$$\boldsymbol{\phi}(u_1, \ldots, u_n) = \left(u_1^{-1}, u_2/u_1, \ldots, u_n/u_1\right). \tag{3.14}$$

Then

$$g(v_1, \ldots, v_n) = v_1 f\left(v_1^{-1}, v_2/v_1, \ldots, v_n/v_1\right) \tag{3.15}$$

defines a convex function on Ω^{ϕ}.

Proof. Since f is convex, it is a supremum of a family affine functions:

$$f(u_1, \ldots, u_n) = \sup_c \left(c_0 + c_1 u_1 + \cdots + c_n u_n\right). \tag{3.16}$$

Since u_1 and v_1 are positive, we may compute:

$$\begin{aligned} g(v_1, \ldots, v_n) &= v_1 f\big(v_1^{-1}, v_2/v_1, \ldots, v_n/v_1\big) \\ &= v_1 \sup_c \left(c_0 + c_1 v_1^{-1} + c_2 v_2/v_1 + \cdots + c_n v_n/v_1\right), \\ &= \sup_c \big(c_0 v_1 + c_1 + c_2 v_2 + \cdots + c_n v_n\big). \end{aligned} \tag{3.17}$$

Thus g is a supremum of affine functions of $v_1, \ldots, v_n$ and is hence convex. □

Since $\mathbf{F} = \mathbf{G}^{-1} = \mathbf{Cof\,G}^T\,|\mathbf{G}|^{-1}$, and $\mathbf{Cof\,F} = \mathbf{F}^{-T}\,|\mathbf{F}| = \mathbf{G}^T\,|\mathbf{G}|^{-1}$, Theorem 3.1 implies that

$$\mathbb{W}^{\phi}\big(\mathbf{G}, \mathbf{Cof\,G}, |\mathbf{G}|\big) = |\mathbf{G}|\,\mathbb{W}\big(\mathbf{Cof\,G}^T\,|\mathbf{G}|^{-1}, \mathbf{G}^T\,|\mathbf{G}|^{-1}, |\mathbf{G}|^{-1}\big) \tag{3.18}$$

is convex, and that E^{ϕ} is a convex function of $\mathbf{v}$, $\mathbf{G}$, $\mathbf{Cof}\,\mathbf{G}$, and $|\mathbf{G}|$.

The energy form is:

$$\begin{aligned} \boldsymbol{\epsilon} &= E\boldsymbol{\rho} - \sum_{\alpha=1}^{3}\sum_{j=1}^{3} v^j \frac{\partial \hat{W}}{\partial \mathbf{F}^j_\alpha} dt \wedge da_{\hat{\alpha}} (-1)^\alpha, \\ &= E\boldsymbol{\rho} - \sum_{i=1}^{3}\sum_{\alpha=1}^{3}\sum_{j=1}^{3} v^j \frac{\partial \hat{W}}{\partial \mathbf{F}^j_\alpha} \left(\mathbf{Cof}\,\mathbf{G}\right)^\alpha_i dt \wedge dx_{\hat{i}} (-1)^i. \end{aligned} \tag{3.19}$$

Thus the Eulerian conservation law for energy is:

$$\begin{aligned} &\frac{\partial E^{\phi}}{\partial t} + \nabla_{\mathbf{x}} \cdot \left(E^{\phi}\mathbf{v}\right) - \sum_{i=1}^{3} \frac{\partial}{\partial x^i} \sum_{\alpha=1}^{3}\sum_{j=1}^{3} v^j \frac{\partial \hat{W}}{\partial \mathbf{F}^j_\alpha} \left(\mathbf{Cof}\,\mathbf{G}\right)^\alpha_i = 0, \text{ or} \\ &\frac{\partial E^{\phi}}{\partial t} + \nabla_{\mathbf{x}} \cdot \left(E^{\phi}\mathbf{v}\right) - \nabla_{\mathbf{x}} \cdot \left(\mathbf{v} \cdot \hat{\mathbf{T}}^{\phi}\right) = 0, \end{aligned} \tag{3.20}$$

For weak solutions with discontinuities in $\mathbf{G}$ or $\mathbf{v}$, we may use the corresponding "entropy condition"

$$\frac{\partial E^{\phi}}{\partial t} + \nabla_{\mathbf{x}} \cdot \left(E^{\phi}\mathbf{v}\right) - \nabla_{\mathbf{x}} \cdot \left(\mathbf{v} \cdot \hat{\mathbf{T}}^{\phi}\right) \leq 0, \tag{3.21}$$

as an admissibility criterion. For non-isothermal motions the conservation of energy equation (3.20) must hold in the sense of distributions, together with (3.13), and the appropriate admissibility criterion is given by the increase of entropy,

$$\frac{\partial S\,|\mathbf{G}|}{\partial t} + \nabla_{\mathbf{x}} \cdot \left(S\,|\mathbf{G}|\,\mathbf{v}\right) \geq 0. \tag{3.22}$$

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