CONSERVATION LAWS, COORDINATE TRANSFORMATIONS, AND DIFFERENTIAL FORMS

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ABSTRACT. It is shown that differential forms and exterior algebra can be very useful in studying weak solutions of hyperbolic systems of conservation laws. Particular examples are given in the area of solid mechanics.

1. INTRODUCTION

There are two different points of view that are used to describe the laws of physics for a continuum. The referential, or Lagrangian, point of view uses a coordinate system which describes points in (material)×(time), whereas the spatial, or Eulerian point of view uses a coordinate system that describes locations in (physical space)×(time). The mapping $(a,t) \to x$ of the material to physical space is called a *motion*. Typically, the motion is an unknown quantity which is determined by the solution of an initial value problem for a system of partial differential equations.

An interesting class of motions consists of those which exhibit shock waves. Mathematically, a shock wave occurs as the result of focussing characteristics for a nonlinear hyperbolic system, and is represented as a discontinuity in the unknowns, as functions of space (or material) and time. When the velocity of the motion $(\frac{\partial x}{\partial t})$ is discontinuous, then the motion will at best be only Lipschitz continuous, or perhaps even discontinuous, as may occur in a shearing motion.

The presence of discontinuities mandates the use of weak solutions to the relevant partial differential equations. The calculus for such solutions is limited. For example, there is no product rule for the weak derivative of a product of L^∞ functions. Thus a question arises, whether the two different coordinate systems are equivalent for such weak solutions. For the case when the map $(a,t) \to (x,t)$ is a bi-Lipschitz homeomorphism, this equivalence has been proved for one space dimension in [7]. The generalization of this result to several space dimensions is straightforward, and is given in [2]. In addition, Dafermos generalized the equivalence to fields in L^1.

In this paper, we will demonstrate some interesting consequences that come from the techniques used in [7, 2]. In particular we will see that the natural language for divergence form conservation laws is the language of differential forms. This language gives an essentially coordinate free description of continuum physics. In addition, this description enables us to make a clear distinction between conservation laws which are physical, and coordinate free, such as laws of conservation of mass, momentum and energy, and those which are mathematical constructs that are not coordinate free, such as equations of continuity.

One result that is particularly interesting is a product rule for weak exterior derivatives. In particular such a product rule holds when α and β are closed forms, i.e., $d\alpha = 0$ and $d\beta = 0$. This

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simple fact enables us to resolve a mystery regarding redundant conservation laws for deformation gradients.

We begin by briefly reviewing the technique that underlies these results. Let t be the time variable, let $\mathbf{a}$ be the referential, or material coordinate, and let $\mathbf{x}$ be the spatial coordinate, so that $\mathbf{a}$ is constant on particle paths. Then $\mathbf{x}(\mathbf{a}, t)$ is the motion of the material. Let

$$\mathbf{v} = \frac{\partial \mathbf{x}}{\partial t} \tag{1.1}$$

be the velocity of the motion, and let

$$F^i{}_\alpha = \frac{\partial x^i}{\partial a^\alpha}. \tag{1.2}$$

The matrix $\mathbf{F}$ is called the deformation gradient of the flow. By equality of mixed partials, which must hold in the sense of distributions, for any Schwarz distribution, we have:

$$\frac{\partial F^i{}_\alpha}{\partial t} = \frac{\partial v^i}{\partial a^\alpha} \tag{1.3}$$

We call this a continuity equation; its purpose is to permit us to forget about the motion as an unknown. Note that (1.3) is not complete in expressing equality of mixed partials. It is shown in [4] however, that (1.3) may be used as an evolution equation for the equality of mixed spatial partial derivatives, as follows. If we differentiate (1.3) with respect to a_β, permute α and β, and subtract, we obtain:

$$\frac{\partial}{\partial t}\left(\left(F^i{}_\alpha\right)_{a^\beta} - \left(F^i{}_\beta\right)_{a^\alpha}\right) = 0 \tag{1.4}$$

Thus, if F^i_α is a gradient in the entire body at $t = 0$, then it is also a gradient for all $t > 0$.

Suppose we have a balance law in material coordinates:

$$\frac{\partial D}{\partial t} + \frac{\partial G_\alpha}{\partial a^\alpha} = H, \tag{1.5}$$

where the repetition of the index α indicates summation from 1 to n, where n is the dimension of the body.

The weak formulation of (1.5) is:

$$\int_0^\infty \int_{\mathbb{R}^n} \phi_t D + \phi_{a^\alpha} G_\alpha + \phi H \; d\mathbf{a}dt = 0. \tag{1.6}$$

This must hold for every smooth test function ϕ with bounded support.

By changing variables from $\mathbf{a}$ to $\mathbf{x}$, we obtain:

$$\int_0^\infty \int_{\mathbb{R}^3} \left(\left(\phi_t + \phi_{x^i} v^i\right) D + \phi_{x^i} F^i{}_\alpha \; G_\alpha + \phi H\right) |\mathbf{F}^{-1}| d\mathbf{x}dt = 0. \tag{1.7}$$

Here, and henceforth, for any $n \times n$ matrix $\mathbf{A}$ we use $|\mathbf{A}|$ to denote the determinant of $\mathbf{A}$.

We regroup:

$$\int_0^\infty \int_{\mathbb{R}^3} \phi_t \left(D|\mathbf{F}^{-1}|\right) + \phi_{x^i} \left(\left(v^i D + F^i{}_\alpha \; G_\alpha\right) |\mathbf{F}^{-1}|\right) + \phi H |\mathbf{F}^{-1}| d\mathbf{x}dt = 0. \tag{1.8}$$

This looks like the weak form of

$$\frac{\partial}{\partial t}\left(D|\mathbf{F}^{-1}|\right) + \frac{\partial}{\partial x^i}\left(\left(v^i D + F^i{}_\alpha \; G_\alpha\right) |\mathbf{F}^{-1}|\right) = H|\mathbf{F}^{-1}|. \tag{1.9}$$

Formally this gives a simple way to transform a balance, or conservation, law from material coordinates to spatial coordinates. The formula for the transformation is the same as the one obtained by performing a change of variables on an integral balance law.

To make this rigorous at the level of distribution theory, we need to state hypotheses under which:

- The change of variables for integrals is valid.
- The Lagrangian test functions $\phi(\mathbf{a},t)$ are mapped into 'enough' Eulerian test functions $\psi(\mathbf{x},t)=\phi(\mathbf{a}(\mathbf{x},t),t)$ so that (1.8) is true for all smooth test functions with compact support.
- The integral in (1.8) makes sense, at least in terms of evaluation of a distribution on a test function ϕ.

A sufficient condition for the above result is that the coordinate transformation $(\mathbf{a},t)\to(\mathbf{x},t)$ is a bi-Lipschitz homeomorphism, and that the fields $(D,F,H)\in L^1_{loc}$ for the material coordinate system. The coordinate transformation is a bi-Lipschitz homeomorphism if there are positive constants C and K such that

$$C|\mathbf{a}_1-\mathbf{a}_2|<|\mathbf{x}(\mathbf{a}_1,\ t)-\mathbf{x}(\mathbf{a}_2,\ t)|<K|\mathbf{a}_1-\mathbf{a}_2|. \tag{1.10}$$

for every t, $\mathbf{a}_1$, and $\mathbf{a}_2$.

Thus the Lagrangian equations for the conservation of mass, momentum, and energy:

$$\begin{aligned}(\rho_{ref})_t&=0\\ \rho_{ref}v^i_t-\left(F^i{}_\alpha S^{\alpha\beta}\right)_{a^\beta}&=f^i\\ \rho_{ref}\left(|\mathbf{v}|^2/2+e\right)_t-\left(v_iF^i{}_\alpha S^{\alpha\beta}\right)_{a^\beta}&=-Q^\beta{}_{a^\beta}+f^iv^i\end{aligned} \tag{1.11}$$

are transformed into the Eulerian equations, where $\rho=\rho_{ref}|\mathbf{F}^{-1}|$

$$\begin{aligned}&\rho_t+(\rho v_j)_{x^j}=0\\ &(\rho v^i)_t+(\rho v^iv^j)_{x^j}-\left(F^i{}_\alpha S^{\alpha\beta}F^j{}_\beta|\mathbf{F}^{-1}|\right)_{x^j}=f^i|\mathbf{F}^{-1}|\\ &\left(\rho\left(|\mathbf{v}|^2/2+e\right)\right)_t+\left(\rho\left(|\mathbf{v}|^2/2+e\right)v^j-\left(v^iF^i{}_\alpha S^{\alpha\beta}F^j{}_\beta|\mathbf{F}^{-1}|\right)\right)_{x^j}+v^if^i|\mathbf{F}^{-1}|\\ &\qquad=-\left(Q^\beta F^j{}_\beta|\mathbf{F}^{-1}|\right)_{x^j}\end{aligned} \tag{1.12}$$

In addition one can obtain other useful 'conservation laws.' For example, the fact that

$$\begin{aligned}\frac{\partial 1}{\partial t}&=0,\\ \frac{\partial 1}{\partial a^\alpha}&=0\end{aligned} \tag{1.13}$$

transforms into equations that seem to have significance:

$$(|\mathbf{F}^{-1}|)_t+(v^i|\mathbf{F}^{-1}|)_{x^i}=0, \tag{1.14}$$

$$(F^i{}_\alpha|\mathbf{F}^{-1}|)_{x^i}=0. \tag{1.15}$$

When ρ_{ref} is constant, (1.14) describes the conservation of mass. The second equation (1.15) is the Piola identity [1]. In addition, the equality of mixed partials,

$$\frac{\partial F^i{}_\alpha}{\partial t}=\frac{\partial v^i}{\partial a^\alpha} \tag{1.16}$$

transforms into a new continuity equation:

$$\left((F^i{}_\alpha|\mathbf{F}^{-1}|\right)_t+\left(\left(F^i{}_\alpha v^j-F^j{}_\alpha v^i\right)|\mathbf{F}^{-1}|\right)_{x^j}=0. \tag{1.17}$$

In some physical problems (elasticity, plasticity) where the stress depends on the components of $\mathbf{F}$, or on the cofactor matrix, $\mathbf{F}|\mathbf{F}|^{-1}$, we need an equation for $\mathbf{F}$, such as (1.17), to close the system; it is a constitutive law. In other areas (gas dynamics) where the stress (pressure) depends on $|\mathbf{F}|$

and not on the components of $\mathbf{F}$, (1.17) is not needed; it is simply a mathematical law regarding the configuration (coordinate transformation) of the gas.

As we remarked concerning (1.3), the equation (1.17) may be understood as an evolution equation. It implies

$$\frac{\partial}{\partial t}\left(\sum_i \left(F^i{}_\alpha |\mathbf{F}^{-1}|\right)_{x^i}\right) = 0, \tag{1.18}$$

so that if the Piola identity holds at $t = 0$, then it must hold for all $t > 0$. Curiously, however, (1.17) does not appear to directly imply an evolution law for the equality of mixed partials in any coordinate system.

An alternative to (1.17, 1.18) was proposed in [6]. This alternative is the equality of mixed partials for $\mathbf{a}(\mathbf{x},\ t)$:

$$\left(F^{-1\alpha}{}_i\right)_t = -\left(F^{-1\alpha}{}_j v_j\right)_{x^i} \tag{1.19}$$

This equation must hold in the weak sense because the inverse map $\mathbf{a}(\mathbf{x},\ t)$ is a well defined Schwarz distribution. This equation may also be treated as an evolution equation, obtaining:

$$\frac{\partial}{\partial t}\left(\left(F^{-1\alpha}{}_i\right)_{x^j} - \left(F^{-1\alpha}{}_j\right)_{x^i}\right) = 0. \tag{1.20}$$

Thus if $\mathbf{F}^{-1}$ is a gradient at $t = 0$, (1.19) implies that it remains a gradient for $t > 0$. $\mathbf{F}^{-1}$ will be a gradient if the initial conditions

$$\left(F^{-1\alpha}{}_i\right)_{x^j}(x,0) = \left(F^{-1\alpha}{}_j\right)_{x^i}(x,0) \tag{1.21}$$

hold.

These two continuity equations (1.17, 1.19) do not appear to be consistent; normally the only transformation one can make on a conservation law, that preserves the jump (Rankine-Hugoniot) conditions is a linear transformation. However, as we have shown, both equations must be true in the weak sense. Do these equations supplement each other, as "extra conservation laws" in the sense of Lax [3], or are they interchangeable? We show, in fact, that they are consistent, although they do not appear to be interchangeable. The Eulerian continuity equation (1.19) implies (1.17); however (1.17) does not appear to imply (1.19). One proof, which we sketch briefly, relies on the coordinate transformation. Equation (1.19), plus suitable initial conditions, implies the existence of $\mathbf{a}(\mathbf{x}, t)$. We then invert this map to obtain $\mathbf{x}(\mathbf{a}, t)$, which in turn implies equation (1.17).

A second proof, which relies on wedge products of differential forms, is more direct. In the author's experience this is the first known example of conservation laws which are not linearly related and still consistent at the level of weak solutions.

2. Second Proof

We begin this discussion by clarifying a mysterious result that was mentioned in Section 1, namely, the equivalence of the trivial equation $\frac{\partial 1}{\partial a^\alpha} = 0$ with the Piola law, (1.15). In the language of differential forms, $\frac{\partial 1}{\partial a^\alpha} = 0$ has the expression $d((-1)^\alpha dt \wedge da_{\hat{\alpha}}) = 0$, in which d is the exterior derivative, and $da_{\hat{\alpha}}$ denotes the alternating product $da^1 \wedge \cdots \wedge da^n$ with the factor da^α omitted. If we formally transform this differential form into (x, t) coordinates, we obtain:

$$(-1)^\alpha dt \wedge da_{\hat{\alpha}} = \sum_{j=1}^n F^j_\alpha |\mathbf{F}|^{-1}(-1)^j dt \wedge dx_{\hat{j}}. \tag{2.1}$$

Equation (2.1) gives two different coordinate expressions for the same differential form. In the material coordinate system, this form is trivially closed. The Piola law expresses the fact that the form remains closed in the physical space-time coordinate system—which is a well known fact for smooth coordinate transformations. Yet, as we saw in Section 1, this must also be true for bi-Lipschitz coordinate transformations.

The foregoing calculation suggests that the wedge product of weakly exact forms is weakly exact. This is particularly interesting, because normally one cannot multiply a weak derivative by a non–smooth function. Yet we shall see that this is indeed the case; the exterior derivative of $\alpha \wedge \beta$ is $d\alpha \wedge \beta \pm \alpha \wedge d\beta$, as long as α, β, $d\alpha$ and $d\beta$ are in suitable L^p_{loc} spaces. Such theorems are also true for partial derivatives of functions, but are less useful for the study of nonlinear hyperbolic conservation laws, because a function with, for example, partial derivatives in L^∞ will be Lipschitz continuous. The exterior derivative does not have this property; indeed, in three space dimensions there are differential forms of degree 1 and 2, with distributional coefficients of any order, whose exterior derivative is zero.

We begin to make these remarks more precise with a few definitions. In particular, we need a definition of weak derivatives for differential forms on a Lipschitz manifold.

We denote by $Sh(p,n)$ the set of injective order preserving maps of $\{1,\dots,p\}$ into $\{1,\dots,n\}$, and for $\sigma \in Sh(p,n)$ we denote by dx_σ the p-form $dx_{\sigma(1)} \wedge \dots \wedge dx_{\sigma(p)}$.

Definition 1. *Let $\omega \in \bigwedge_p(U)$, $U \subset \mathbb{R}^n$. Let $x_1,\dots,x_n$ be coordinates on $\mathbb{R}^n$, and suppose that $\omega = \sum_{\sigma\in Sh(p,n)} \omega_\sigma(x) dx_\sigma$.*

1. *Let $1 \le q \le \infty$. We say that $\omega \in L^q_{loc}\left(U, \bigwedge_p\right)$ if each coefficient ω_σ is in $L^q(K)$ for each compact set $K \subset U$.*
2. *If $\omega \in L^q_{loc}\left(U, \bigwedge_p\right)$, for some $q \ge 1$, we say that $d\omega$ is the unique linear functional T on the space $Lip_0\left(U, \bigwedge_{(n-p-1)}\right)$ defined by $T(\beta) = \int_U (-1)^{p-1} \omega \wedge d\beta$, for every Lipschitz continuous $(n-p-1)$-form β, with compact support. Note that $d\beta$ will be defined almost everywhere, by Rademacher's theorem, and that $d\beta \in L^\infty_{loc}$.*

The next theorem shows that these definitions are invariant under bi-Lipschitz coordinate transformations.

Theorem 2.1. *Let ω be a locally integrable p–form on $U \subset \mathbb{R}^n$. Let $T : V \to U \subset \mathbb{R}^n$ be a bi-Lipschitz homeomorphism. Then*

1. *$T^*\omega$ is a locally integrable p–form on V.*
2. *$d(T^*\omega) = T^* d\omega$.*

Proof. (1) If $\omega = \sum_{\sigma\in Sh(p,n)} \omega_\sigma(x) dx_\sigma$, then

$$T^*\omega = \sum_{\sigma\in Sh(p,n)} \omega_\sigma(T(y)) \left(\sum_{j=1}^n \frac{\partial T_{\sigma(1)}}{\partial y_j} dy_j\right) \wedge \dots \wedge \left(\sum_{j=1}^n \frac{\partial T_{\sigma(p)}}{\partial y_j} dy_j\right) \tag{2.2}$$

Let K be a compact subset of V. Then $T(K)$ is a compact subset of U, and ω is integrable on $T(K)$. Then for any $\gamma \in Sh(p,n)$, the coefficient of $T^*\omega$ corresponding to dy_γ is a linear combination of terms of the form

$$\omega_\sigma(T(y)) \prod_{k=1}^p \frac{\partial T_{\sigma(k)}}{\partial y_{\gamma(k)}} \tag{2.3}$$

Since T is bi-Lipschitz, the factors $\frac{\partial T_{\sigma(k)}}{\partial y_{\gamma(k)}}$ are in L_∞, as is the Jacobian $|DT|^{-1}$ of T^{-1} Thus

$$\int_K |\omega_\sigma(T(y)) \prod_{k=1}^p \frac{\partial T_{\sigma(k)}}{\partial y_{\gamma(k)}}| dy = \int_{T(K)} |\omega_\sigma(x) \prod_{k=1}^p \frac{\partial T_{\sigma(k)}}{\partial y_{\gamma(k)}}| |DT|^{-1} dx. \tag{2.4}$$

Thus the coefficients of dy_γ are integrable on K.

(2) Let β be a Lipschitz continuous differential form of degree $n-p-1$, and of compact support. Then $T^{-1*}\beta$ is a Lipschitz continuous differential form on U, and $d\left(T^{-1*}\beta\right) = T^{-1*}d\beta$. Thus

$$\int_V (-1)^{p-1} T^*\omega \wedge d\beta = \int_U (-1)^{p-1}\omega \wedge d\left(T^{-1*}\beta\right) \tag{2.5}$$

$$= \int_U d\omega \wedge T^{-1*}\beta \tag{2.6}$$

$$= \int_V T^*\left(d\omega\right) \wedge \beta \quad \square \tag{2.7}$$

Next we define smoothing for differential forms, in a way that respects the exterior derivative.

Definition 2. *Let ω be a locally integrable p–form on $U \subset \mathbb{R}^n$. Extend ω as a p–form on $\mathbb{R}^n$ by defining $\omega(x) = 0$ if $x \notin U$. Let $\omega(x) = \sum_{\sigma \in Sh(p,n)} \omega_\sigma(x) dx_\sigma$. Let j be a smooth function with compact support in $\mathbb{R}^n$ such that $\int_{\mathbb{R}^n} j(x)dx = 1$. Then for each $\sigma \in Sh(p,n)$ and each $\epsilon > 0$, define*

$$\omega_{\sigma\epsilon}(x) = \int_{\mathbb{R}^n} \omega_\sigma(x-y) j\left(\frac{y}{\epsilon}\right)\epsilon^{-n} dy. \tag{2.8}$$

Finally, let $\omega_\epsilon = \sum_{\sigma \in Sh(p,n)} \omega_{\sigma\epsilon} dx_\sigma$.

We note that ω_ϵ is smooth in the x-coordinate system, and that locally Lipschitz coordinate transformations will transform ω_ϵ as a locally Lipschitz form. If $\omega \in L^p_{loc}$, for $1 \le p < \infty$, then ω_ϵ converges to ω in L^p_{loc} as $\epsilon \to 0$. Also, one may readily verify that $d(\omega_\epsilon) = (d\omega)_\epsilon$. Note that although our definition of smoothing is coordinate dependent, commutation of smoothing with d is coordinate-free.

Our main theorem is the following.

Theorem 2.2. *Let $\alpha \in L^p_{loc}$ and $\beta \in L^{p'}_{loc}$ be differential forms of degree q and q', respectively, where $\frac{1}{p} + \frac{1}{p'} = 1$, and further suppose that $d\alpha \in L^p$ and $d\beta \in L^{p'}$. Then $d(\alpha\wedge\beta) = d\alpha\wedge\beta + (-1)^q \alpha \wedge d\beta$, and is locally integrable.*

Proof. We approximate α and β by smooth forms α_ϵ and β_ϵ, as in Definition 2. Then $d\left(\alpha_\epsilon \wedge \beta_\epsilon\right) = d\alpha_\epsilon \wedge \beta_\epsilon + (-1)^q \alpha_\epsilon \wedge d\beta_\epsilon$. As $\epsilon \to 0$, we have for each compact set $K \subset \mathbb{R}^n$, using the bilinearity of the wedge product together with the triangle inequality,

$$||d\alpha_\epsilon \wedge \beta_\epsilon - d\alpha \wedge \beta||_{1,K} \le ||d\alpha_\epsilon - d\alpha||_{p,K} ||\beta_\epsilon||_{p',K} + ||d\alpha||_{p,K} ||\beta_\epsilon - \beta||_{p',K} \to 0, \tag{2.9}$$

$$||\alpha_\epsilon \wedge d\beta_\epsilon - \alpha \wedge d\beta||_{1,K} \to 0, \tag{2.10}$$

and we conclude by the completeness of $L^1(K)$ that $d\left(\alpha\wedge\beta\right) \in L^1(K)$, and

$$d\left(\alpha \wedge \beta\right) = d\alpha \wedge \beta + (-1)^q \alpha \wedge d\beta. \quad \square \tag{2.11}$$

Theorem 2.2 is not new (see, for example, [5], eq. 11). However the applications which we now present for this theorem are new and interesting. First, Theorem 2.2 implies that our formal calculation for the Piola identity is valid whenever, for example, the differential forms $da_\alpha, \alpha = 1, \ldots, n$ are in L_∞, and are closed, that is, equations (1.19, 1.21) hold. Thus equations (1.19, 1.21) imply the Piola identity, even for weak solutions.

Similar calculations may be performed with ease and simplicity. For example, we noted that the conservation of mass, (1.14) was implied by the Lagrangian equation $\frac{\partial 1}{\partial t} = 0$. In terms of differential forms, this says that

$$da_1 \wedge \ldots \wedge da_n = |\mathbf{F}^{-1}| dx_1 \wedge \ldots \wedge dx_n + v^i |\mathbf{F}^{-1}| (-1)^i dt \wedge dx_{\hat{i}}. \tag{2.12}$$

Thus, even for weak solutions, (1.14) is implied by (1.19, 1.21).

The cofactor equation, (1.17), was obtained by transforming (1.3) as a divergence form conservation law, that is, we transformed the underlying 1-form, $dx_i = v^i dt + \sum_{\alpha=1}^{3} F^i_\alpha da_\alpha$, as a form of degree 3. This was accomplished by forming the product $dx_i \wedge da_{\hat{\alpha}}$ and then expressing the result in Eulerian (x) coordinates. Thus, if we take the Eulerian expression which says that $dx_i \wedge da_{\hat{\alpha}}$ is closed, we obtain (1.17).

Of course, the natural way to express (1.3) in Eulerian coordinates is with the trivial equation $d(dx_i) = 0$. Similarly, it is natural to transform (1.19) as $d(da_\alpha) = 0$. The cofactor equations, while they do not appear to be natural continuity equations, may be useful for other purposes.

From these observations, it is apparent that a continuity equation belongs to a particular coordinate system, whereas physical conservation laws, such as conservation of mass, momentum and energy, are coordinate free. However the conservation of mass becomes trivial in a Lagrangian coordinate system. It may be somewhat distressing to think of conservation of mass as trivial, but in Lagrangian coordinates (the reference configuration), conservation of mass is built in to the coordinate system. Instead one must be concerned about the conservation of volume, as expressed by the density of volume with respect to mass.

Other conservation laws may be derived using these techiniques. For example, we can show that the characteristic polynomial of $\mathbf{F}$ is conserved, as follows. Let $\omega_i = dx_i - \lambda\delta^i_\alpha da_\alpha$; ω_i is the differential of $y_i = x_i - \lambda a_\alpha$. We can think of y_i as a new configuration of the material. The corresponding deformation gradient is $\mathbf{F} - \lambda\mathbf{I}$, and the corresponding velocity is $\mathbf{v}$. Then

$$d\omega_1 \wedge d\omega_2 \wedge d\omega_3 = |\mathbf{F} - \lambda\mathbf{I}| da_1 \wedge da_2 \wedge da_3 - \sum_{\alpha=1}^{3} \left((\mathbf{F} - \lambda\mathbf{I})^{-1} \mathbf{v} \right)_\alpha dt \wedge da_{\hat{\alpha}} (-1)^\alpha \tag{2.13}$$

From this one can derive conservation laws for each of the coefficients of the characteristic polynomial of $\mathbf{F}$.

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