

Option price sensitivity to errors in stochastic dynamics modeling

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Abstract

When asset prices are modelled by stochastic dynamics, the model parameters are estimated from financial data. We study how estimation errors on model parameters impact the computed option prices, in the case where asset price and volatility follow the classical joint SDEs parametric model of Heston. Model parameters are estimated by an approximate maximum likelihood approach studied in [4] which presented an implementable computation of the covariance matrix for the errors in model parameters estimation. We then study and compute the sensitivity of optimal option prices to errors on the model parameters. This is achieved by numerically solving the PDEs verified by the derivatives of the option price with respect to model parameters. Combining these evaluations of derivatives with the computed covariance matrix of errors on model parameters, we obtain the errors on option price due to parametric estimation errors. We apply our method to the S&P 500 index options using VIX [29] as a proxy for volatility.

1 Introduction

It is common practice in finance applications to model the price of securities and other assets using stochastic differential equations. The models are estimated using available data and are then used for forecasting or for computing the price of financial instruments.

We study the impact of errors in model estimation on the price of options where the underlying asset is assumed to verify the estimated model. We study how errors on each individual parameter affect option prices. We compute a bound on the error in the option price due to model error.

Extensions of the Black-Scholes models to local volatility ([8],[9]) and stochastic volatility models ([19],[25]) are often used in practice. Local volatility models allow for the instantaneous volatility to be a deterministic function of time and the asset price. In stochastic volatility models the volatility is driven by a diffusion process. We focus our study on the classical Heston stochastic volatility models, driven by two cou-

pled SDEs.

Volatilities are not directly observed in the market, and the transition density of the coupled process is not available in closed form. We estimate the Heston model parameters using an efficient approximate maximum likelihood method introduced in [4], where variances of these parameter estimators were also computed. Here we outline the numerical computation of the derivatives of option prices with respect to each model parameter by writing and solving associated parabolic PDEs. We also show how to compute derivatives of the option price with respect to the so called “market price” of volatility risk.

We apply our methods to the case of S&P 500 index options where we use VIX as a proxy for volatility.

2 Stochastic volatility model

Consider the price process $\{X_t\}$ of a tradable asset (stock or index). Let V_t be its instantaneous volatility and $Y_t = V_t^2$. Following Heston [19] we assume that $\{X_t\}$ and $\{Y_t\}$ follow a pair of correlated diffusion processes given by,

$$(2.1) \quad dX_t = \mu X_t dt + \sqrt{Y_t} X_t d\bar{W}_1(t)$$

$$(2.2) \quad dY_t = \kappa(\theta - Y_t) dt + \gamma \sqrt{Y_t} d\bar{W}_2(t)$$

where $\{\bar{W}_1(t), \bar{W}_2(t)\}$ is a pair of correlated Brownian motions with $E[d\bar{W}_1(t), d\bar{W}_2(t)] = \rho dt$ ($|\rho| < 1$).

In particular the logarithm of the stock price $L_t = \ln(X_t)$ verifies

$$(2.3) \quad dL_t = (\mu - \frac{1}{2}Y_t) dt + \sqrt{Y_t} d\bar{W}_1(t)$$

The parameter μ is the mean rate of return of the asset price. For parameters $\kappa, \theta > 0$ (2.2) is the well known mean reverting square root process ([12],[7]) originally used to model short term interest rates. The process Y_t is pulled towards the long run mean θ with speed of mean reversion κ . The parameter γ is the instantaneous relative volatility of $\{Y_t\}$. The transition density of $\{Y_t\}$ is a non central chi-square and its stationary distribution is a gamma distribution with mean θ and variance $\frac{\gamma^2 \theta}{2\kappa}$.

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We assume that $2\kappa\theta > \gamma^2$, a classical condition under which the upward drift in (2.2) keeps $\{Y_t\}$ positive for all t [12] when $\{Y_0\} > 0$. Note that for $2\kappa\theta = \gamma^2$, (2.2) is an Ornstein-Uhlenbeck process.

2.1 Model identification To identify parameters in the model (2.1)-(2.2) for best fit with a set of available jointly observed data, [1] used a Hermite approximation of the likelihood, [2] developed a weighted non parametric approach to determine the risk-neutral measure of future market states, [22] proposed a Bayesian estimation.

Here we use a very efficient estimation technique we had introduced in [4], based on Euler discretization of the joint SDEs model; we solve first a constrained optimization problem to identify the parameters in (2.2). The parameter μ is then readily obtained by maximum likelihood based on (2.1) only. Once the parameters $\mu, \kappa, \theta, \gamma$ are estimated we reconstruct the underlying Brownian motions and then directly estimate their correlation coefficient ρ . This decoupling of the SDEs does not reduce the accuracy of the parameter estimates as compared to fully joint estimation, and provides a strong numerical gain, since all estimators can be explicitly computed. The natural domain for the unknown parameters is defined by the following constraints,

$$\kappa, \theta, \gamma > 0, \quad 0 < \theta < 1, \quad 2\kappa\theta > \gamma^2, \quad -1 \leq \rho \leq 1$$

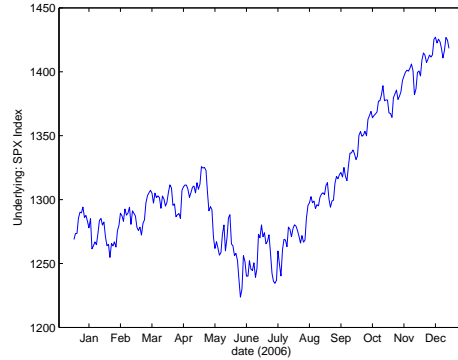
We now present our parameter identification for the modelling of observed S&P 500 index data.

2.2 Estimation of a Heston model for (SPX,VIX) data We estimate the Heston stochastic volatility model for the S&P 500 index data (SPX). We use the VIX index as a proxy for volatility, since the VIX values represent the implied volatility of a group of options on the S&P 500, short dated at the money, and is officially computed by the CBOE [29].

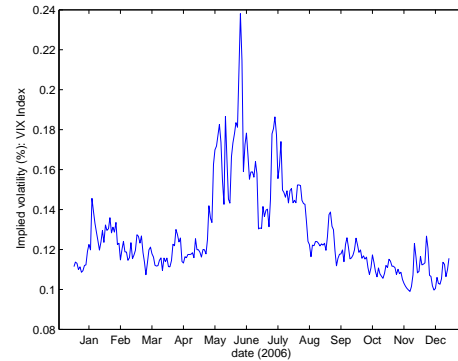
We use 252 daily S&P 500 index data from Jan 03 to Dec 29 2006. To maintain the simultaneity of SPX and VIX we use the closing data for both. Following standard practice, the time step Δ between two consecutive daily (SPX,VIX) observations is equal to $\frac{1}{252}$. The evolution of SPX and VIX over this period is displayed in Fig.1(a) and Fig.1(b).

The parameter values are estimated by the algorithm just outlined above [4] and displayed in Table(1) Col. 2.

2.3 Evaluation of estimation error We compute the mean bias, variances, and relative errors of our parameter estimators. Here relative error is the ratio



(a) The SPX (S&P 500) index is the asset price value $\{X_t\}$



(b) The VIX index represents the volatility value for options on the S&P 500 index.

Figure 1: Asset price and volatility

of standard deviation of the estimator to the true parameter value. To compute these quantities we simulate 1000 trajectories of length 252 for the Heston joint SDEs with the estimated parameter values as true values and then obtain the mean bias, variance and relative error over the 1000 trajectories. Columns 3, 4,5 of Table(1) display these quantities.

The maximum likelihood estimates that we obtain from the Euler discretization of the model are asymptotically Gaussian and we have [3],

$$\lim_{N \rightarrow \infty} cov(\sqrt{N}(\hat{\Theta} - \Theta)) = \Sigma(\Theta)$$

where Θ is the vector of model parameters and N is the number of observations. An estimate of the asymptotic value of the variance-covariance matrix $\Sigma(\Theta)$ for the S&P 500 dataset is given below. This was computed by simulating 500 trajectories of the 2 joint Heston SDEs,

Parameter	Estimate	Bias	Variance	Relative Error
κ	16.6031	1.0017	8.26	16%
θ	.0170	.0003	5.3e-06	13%
γ	.2826	-.0017	.0002	5%
ρ	-.5441	.0036	.0035	10%
μ	.1349	.0333	.007	60%

Table 1: Estimated parameters for a Heston model fitted to the (SPX,VIX) one year dataset ($N = 252$). Column 2 = parameter estimates. Column 3 = Mean Estimator Bias, Column 4 = Estimator Variance, Column 5 = standard deviation / true value.

of length $N = 10000$.

$$\Sigma = \begin{pmatrix} 6856 & -.8730 & .3226 & -1.6142 \\ -.8730 & .0011 & 0 & 0 \\ .3226 & 0 & .0430 & -.0867 \\ -1.6142 & 0 & -.0867 & .8004 \end{pmatrix}$$

We refer to [4] for explicit formulas giving the asymptotic variance of the maximum likelihood estimates based on a discretization of the Heston model.

From Fig.1 we see that when the value of the S&P 500 index falls the value of the VIX index increases. This is explained by the negative correlation coefficient (ρ) for this (SPX,VIX) dataset. This is referred to as the *leverage effect* in financial literature.

As is known from the no arbitrage pricing theory the risk neutral value of the option price does not depend directly on the parameter μ . Therefore we will not need to include this parameter in our sensitivity analysis. We will refer to,

$$PAR = \{\kappa, \theta, \gamma, \rho\}$$

as the vector of four key model parameters for our sensitivity study. Then $PAR \in \Omega_{PAR}$ where

$$(2.4) \quad \Omega_{PAR} = \{PAR \in \mathbb{R}^4 : \kappa, \theta, \gamma > 0, \gamma^2 < 2\kappa\theta, |\rho| < 1\}$$

For parameter fitting of the model (2.1)-(2.2) to given sets of daily data, one has to fix a time unit for the variable “ t ”, which is equivalent to selecting a value Δ for the time period between two successive daily observations. Usually one sets $\Delta = 1/252$ for daily data, but we point out that the relative accuracy of the maximum likelihood parameter estimates is independent of the choice of Δ when relative accuracy of an estimate is defined by the ratio of standard deviation over mean.

3 Option pricing

A European call (resp. put) option is a financial contract which gives its holder the right to buy (resp.

sell) the underlying asset, at a future fixed “maturity” date T , at a predetermined “strike price” K , fixed at the time the option is created. The option holder can exercise his right to buy (or sell) at maturity only. However the option can be traded at anytime between its creation and maturity.

The European call option with maturity T is characterized by its payoff function,

$$\Phi(X_T) = (X_T - K)^+ = \max(X_T - K, 0)$$

Let $\{Z_t, 0 \leq t \leq T\}$ be the price of an European call option with maturity T . It is standard to assume that under stochastic volatility the price Z_t is a function of X_t and Y_t [19],

$$Z_t = f(X_t, Y_t, t)$$

The theory of no arbitrage pricing implies that the option price is computed under the so called equivalent martingale measure (EMM). The discounted price of any traded asset is a martingale under this measure. The dynamics of $\{X_t, Y_t\}$ under the EMM is given by [18],

$$(3.5) \quad dX_t = rX_t dt + \sqrt{Y_t}X_t dW_1(t)$$

$$(3.6) \quad dY_t = \kappa((\theta - Y_t) - \lambda_t \gamma \sqrt{Y_t})dt + \gamma \sqrt{Y_t} dW_2(t)$$

The dynamics for the log of the stock price L_t under the risk neutral measure is given by,

$$(3.7) \quad dL_t = (r - \frac{1}{2}Y_t)dt + \sqrt{Y_t}dW_1(t)$$

where $E[dW_1(t), dW_2(t)] = \rho dt$. The parameter r is a risk free short term rate of return assumed to be constant, for instance short term yield of U.S. treasury bonds.

The random variable λ_t is a function of the market price of volatility risk and arises due to the non tradability of the volatility. The market is said to be incomplete in this case. For equity options [23] presents evidence of non zero market price of volatility risk.

Define,

$$g(X_t, Y_t, t) = f(X_{T-t}, Y_{T-t}, T - t), \quad 0 \leq t \leq T$$

Then g verifies the following parabolic PDE ([5],[19]),

$$(3.8) \quad \frac{\partial g}{\partial t} - Lg = 0 \text{ on } (0, \infty) \times (0, \infty) \times (0, T]$$

with initial condition,

$$g(x, y, 0) = (x - K)^+ \text{ on } (0, \infty) \times (0, \infty) \times \{t = 0\}$$

where L is the linear operator,

$$L = rx \frac{\partial}{\partial x} + [\kappa(\theta - y) - \lambda\sqrt{y}\gamma] \frac{\partial}{\partial y} + \frac{1}{2}x^2y \frac{\partial^2}{\partial x^2} + \frac{1}{2}\gamma^2y \frac{\partial^2}{\partial y^2} + \rho\gamma xy \frac{\partial^2}{\partial x \partial y} - r$$

We have assumed λ_t to be constant similarly to [13]. We now describe the boundary conditions satisfied by the option price [19]. The boundary conditions remain unchanged after the transformation in the time variable. When the asset price is zero there is no rational interest in buying the call option and therefore the option is worthless,

$$g(0, y, t) = 0 \text{ on } \{x = 0\} \times (0, \infty) \times (0, T)$$

For large values of the asset the option price grows linearly with the asset price,

$$\lim_{x \rightarrow \infty} \frac{\partial g(x, y, t)}{\partial x} = 1, \forall y, t$$

For large values of Y_t the option price tends to be constant as a function of the square of volatility,

$$\lim_{y \rightarrow \infty} \frac{\partial g(x, y, t)}{\partial y} = 0, \forall x, t$$

The boundary condition at $y = 0$ is obtained by setting y to be zero in the PDE (3.8),

$$(3.9) \quad \frac{\partial g}{\partial t} - rx \frac{\partial g}{\partial x} - \kappa\theta \frac{\partial g}{\partial y} + rg = 0,$$

on $(0, \infty) \times \{y = 0\} \times (0, T]$. For all practical applications we can consider the problem on the bounded domain,

$$U_T = U_B \times (0, T]$$

where

$$U_B = (0, X_{max}) \times (0, Y_{max})$$

The option pricing problem is a two dimensional second order parabolic partial differential equation where the operator L is elliptic for each t in x and y . More precisely it is a convection-diffusion type equation with Dirichlet boundary at $x = 0$ and Neumann boundary as x and y approach X_{max} and Y_{max} respectively [17]. The boundary condition at $y = 0$ is not of a standard type. We refer to (3.8) together with the initial and boundary conditions as an initial/boundary-value problem [11]. [10] shows that the option price given by the stochastic representation formula is a classical solution of the PDE (3.8) and satisfies the boundary condition at vanishing volatility.

3.1 Estimation of the market price of volatility risk λ

We need an estimate of λ in order to solve the pricing equation. According to the no arbitrage theory λ should be estimated from one volatility dependent asset and used to price all other volatility dependent assets. [13] proposes an estimate of a constant λ for stochastic volatility models using observed option prices. As suggested in [19] one could estimate the parameter λ using average returns on option positions that are hedged against the risk of changes in the asset price.

In a parallel paper [30], we introduced and studied a new algorithm for the estimation of the market price λ of volatility risk for a tradable asset X_t , derived from estimated time dependent Black-Scholes approximations of the observed dynamics of several options based on this asset. We have applied this method to the (SPX,VIX) dataset to get an estimate of the corresponding λ .

From a group of short maturity options with strike price close to the prevailing SPX values we get thus an estimate $\lambda = 2$, validated by simultaneously comparing the mean squared error in time between predicted and actual option prices for a range of different λ values. We report concrete results here for one specific European option EOP_t with strike price 1430 and maturity date Feb 17 2007, observed daily from Jan 03 2007 to Feb 02 2007 .

We used the estimated model parameters of Table(1) for the dynamics of the asset price, and we set the value of the risk free rate of return at $r = .01$. Using the estimated $\lambda = 2$, we solved the option price PDE (3.8) to compute dynamic estimates $E\hat{O}P_t$ of the option price EOP_t , and we derived the mean value 0.66 and standard deviation 0.96 of the prediction error $E\hat{O}P_t - EOP_t$ for the option price, which had a mean value of 15.63 during this period. Fig.2 represents the computed and market price of this European option EOP_t .

To study, in the next sections, how sensitive the option price is to the model parameters PAR and the impact of model estimation errors on the option price, we fixed the value $\lambda = 2$ just estimated from market data.

4 Impact of model estimation errors on option price

In this section we compute a bound on the error in the option price generated by random errors on parameters of the joint Heston SDEs. These random errors are due to the estimation of the model parameters from observed data on asset price and volatility. The option price $g(x, y, t)$ as a solution of the PDE (3.8), depends on all the model parameters except μ . We will use both

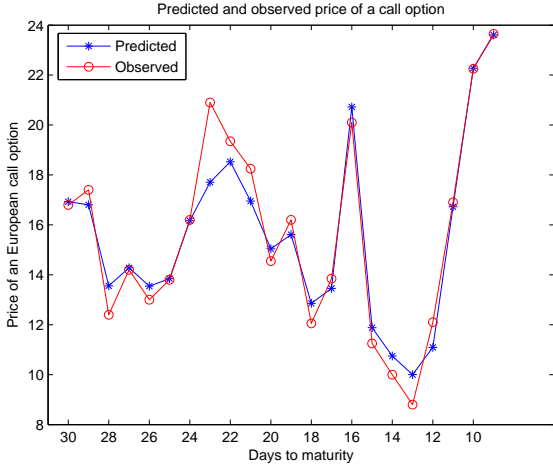


Figure 2: Predicted and observed option price of an European call option with strike price 1430 for $\lambda = 2$

an extended and a shorthand notation for g , defined by

$$g(x, y, t; PAR, \lambda) = g(PAR) = g(x, y, t)$$

where $PAR = (PAR_i)_{i=1}^4$ is the vector,

$$PAR_1 = \kappa, \quad PAR_2 = \theta, \quad PAR_3 = \gamma, \quad PAR_4 = \rho$$

It can be proved that $g(PAR)$ is a smooth function of the parameter vector PAR in the domain Ω_{PAR} defined above by (2.4).

Let P be the true but unknown value of the parameter vector PAR of the model (2.1)-(2.2). We expect P to be in a neighborhood of the estimated parameter $Q = \hat{P}$. For all $i = 1, 2, 3, 4$ define,

$$D_i(g)(x, y, t; PAR, \lambda) = \frac{\partial g}{\partial PAR_i}(x, y, t; PAR, \lambda)$$

A first order Taylor expansion of $g(PAR)$ at $PAR = P$ yields,

$$(4.10) \quad g(Q) \simeq g(P) + \sum_{i=1}^4 D_i(g(P)) \cdot (Q_i - P_i),$$

We use the following notations:

$$D_{i,P} = D_i(g(P)),$$

$$Cov_{i,j}(P) = Cov[\sqrt{N}(Q_i, Q_j)],$$

$$\sigma_i^2(P) = Var[\sqrt{N}(Q_i - P_i)]$$

and the mean quadratic error on the option price

$$\delta(P)^2 = E[(g(Q) - g(P))^2],$$

Here N denotes the number of actual daily observations of the pair (asset price, volatility) used to compute the estimated parameter vector Q .

Squaring both sides of (4.10) and taking the expectation with respect to the distribution of the estimator Q yields

$$(4.11) \quad \delta(P)^2 = E[(g(Q) - g(P))^2] \simeq \sum_{i=1}^4 \frac{\sigma_i^2(P)}{N} D_{i,P}^2 + 2 \sum_{1 \leq i, j \leq 4, i \neq j} \frac{Cov_{i,j}(P)}{N} D_{i,P} D_{j,P}.$$

The true P is unknown, but given the estimated Q , we expect P to be in a ‘‘confidence neighborhood’’ $B(Q) \subset \Omega_{PAR} \subset \mathbb{R}^4$ of Q . Therefore, a confidence upper bound on the L^2 -error $\delta(P)$ in option price will be the maximum of $\delta(P)$ over $B(Q)$. Using $\varepsilon(Q)$ to denote the confidence bound in L^2 for the error $g(Q) - g(P)$ on the option price at Q we then can take,

$$\varepsilon(Q) = \max_{P \in B(Q)} \delta(P)$$

For large N , equation (4.11) only requires the parameter estimates to be unbiased. This is true for our maximum likelihood parameter estimates. It follows from (4.11) that when parameter estimates are asymptotically Gaussian, the first order error terms go to zero in L^2 norm as N goes to infinity.

In our practical applications, we define the confidence neighborhood $B(Q)$ by

$$B(Q) = B_1 \times B_2 \times B_3 \times B_4$$

with

$$B_i = [Q_i - \hat{\sigma}_i, Q_i + \hat{\sigma}_i].$$

where $\hat{\sigma}_i$ is an estimate of the standard deviation of Q_i . Depending on the number of observations and the accuracy of estimates we can expand the sets B_i to a radius of $2\hat{\sigma}_i$. We then naturally define the *sensitivity* of the option price $g(PAR)$ with respect to each parameter PAR_i as the product of the derivative of $g(PAR)$ with respect to PAR_i multiplied by the standard deviation of PAR_i , with the following notations

$$Sen_\kappa = |D_{1,P}| \sigma_\kappa, \quad Sen_\theta = |D_{2,P}| \sigma_\theta, \quad Sen_\gamma = |D_{3,P}| \sigma_\gamma,$$

$$Sen_\rho = |D_{4,P}| \sigma_\rho$$

where $\sigma_\kappa, \sigma_\theta, \sigma_\gamma$ and σ_ρ are the respective standard deviations of the estimators of $\kappa, \theta, \gamma, \rho$.

We hence proceed to compute the derivatives of the option price with respect to the model parameters.

5 Differentiability of the option price with respect to model parameters

Before computing sensitivities of the option price to parameter errors, we review the question of existence, uniqueness, and differentiability (with respect to parameters) for the solution of the parabolic PDE and boundary value problem defining the option price.

The extensive literature on parabolic PDEs does not seem to cover explicitly the specific initial/boundary value problem that we have here; see for instance the semi-group method [28] or the weak sense solutions approach [26]. So we sketch here a proof for the differentiability of the option price with respect to its parameters. The option price is a function of the logarithm lsp of the stock price, denoted by

$$h(lsp, y, t) = f(\exp(lsp), y, t) \text{ where} \\ t \in (0, T), l \in (-\infty, \infty), y \in (0, \infty)$$

Let $Z_t = (L_t, Y_t)$. We will also use the shorthand notations $z = (lsp, y)$ and $h(lsp, y, t) = h(z, t)$. A probabilistic expression for the option price at time t from the no arbitrage pricing theory is given by [18],

$$(5.12) \quad h(z, t) = e^{-r(T-t)} E(\phi(Z_T) | Z_t = z)$$

where $z = (lsp, y)$. Recall the definition of the payoff function in terms of lsp ,

$$\phi(z) = \phi(lsp, y) = \max(\exp(lsp) - K, 0)$$

The conditional expectation of the random variable $\phi(Z_T)$ given $Z_t = z$ in (5.12) is computed under the risk neutral measure. Let

$$u(t) = \exp(-r(T-t)).$$

Then,

$$h(z, t) = u(t) E(\phi(Z_T) | Z_t = z) \\ (5.13) \quad h(z, t) = u(t) \int_{\Omega} q(t, z; T, Z) \phi(Z) dZ$$

where $q(t, z; T, Z)$ is the conditional density of the random variable Z_T given $Z_t = z$. The set Ω is the spatial domain,

$$\Omega = \mathbb{R} \times \mathbb{R}^+$$

The transition density function $q(t, z; T, Z)$ is known to satisfy the Kolmogorov forward equation as a function of T and Z [15],

$$(5.14) \quad \left(\frac{\partial}{\partial T} - \Delta\right)q = 0 \text{ on } (t, \infty) \times \mathbb{R} \times \mathbb{R}^+$$

with

$$\lim_{T \rightarrow t} q(t, z; T, Z) = \delta_z$$

where

$$\Delta = - \langle b, \nabla \rangle + \frac{1}{2} \text{tr}(H\sigma\sigma^*)$$

where b is the column vector of drift coefficients and σ is the column vector of diffusion coefficients of (3.7) and (3.6). ∇ is the gradient vector and $\text{tr}(H\sigma\sigma^*)$ is the trace of the matrix $H\sigma\sigma^*$ where H is the Hessian matrix. In the equation above we have omitted the dependence of q on the state and time variables.

It is well known that the solution q of the forward equation is a smooth function of the parameters in (5.14) ([14], [20]). To compute the derivative of the option price with respect to the model parameters we need to differentiate with respect to the parameters under the integral sign on the right hand side of equation (5.13). Since the function ϕ does not depend explicitly on the parameters, one only needs to verify that the derivatives of q with respect to the parameters are integrable in the domain Ω . This requires deriving good upper bounds on the derivatives at lsp, y, t when y tends to zero, which can be done by applications of the maximum principle. We have in any case verified numerically that the evaluations of the derivatives obtained by solving the formal PDEs generated by differentiating the PDE for $g(\text{PAR})$ with respect to each parameter do converge to the derivative of the option price with respect to that parameter.

We now study the sensitivity of the option price with respect to λ also.

Define the vector $\mathbf{p} \in \mathbb{R}^5$ by $\mathbf{p}_i = \text{PAR}_i$ for $i = 1, 2, \dots, 4$ and $\mathbf{p}_5 = \lambda$. For all $i = 1, 2, \dots, 5$ define,

$$D_i(g)(x, y, t; \mathbf{p}) = \frac{\partial g}{\partial p_i}(x, y, t; \mathbf{p})$$

Differentiating (3.8) with respect to each of the parameters p_i the *sensitivity PDEs* take the following form,

$$(5.15) \quad \left(\frac{\partial}{\partial t} - L\right)D_i = G_i(x, y, t; \mathbf{p}) \text{ on } U_T$$

with initial condition

$$(5.16) \quad D_i(x, y, 0; \mathbf{p}) = 0 \text{ on } U_B \times \{t = 0\}$$

We see that the sensitivity equations are two dimensional non homogeneous parabolic partial differential equations where the right hand side is given by,

$$G_1(x, y, t; \mathbf{p}) = (\theta - y) \frac{\partial g}{\partial y}$$

$$G_2(x, y, t; \mathbf{p}) = \kappa \frac{\partial g}{\partial y}$$

$$G_3(x, y, t; \mathbf{p}) = \gamma y \frac{\partial^2 g}{\partial y^2} + \rho xy \frac{\partial^2 g}{\partial x \partial y} - \lambda \sqrt{y} \frac{\partial g}{\partial y}$$

$$G_4(t, x, y; \mathbf{p}) = \gamma xy \frac{\partial^2 g}{\partial x \partial y},$$

$$G_5(t, x, y; \mathbf{p}) = -\gamma \sqrt{y} \frac{\partial g}{\partial y}$$

The boundary conditions for the sensitivity PDEs are,

$$D_i = 0, \text{ on } \{x = 0\} \times (0, Y_{max}) \times (0, T]$$

$$\lim_{x \rightarrow X_{max}} \frac{\partial D_i}{\partial x} = 0, \forall y, t$$

$$\lim_{y \rightarrow Y_{max}} \frac{\partial D_i}{\partial y} = 0, \forall x, t$$

for all $i = 1, 2, \dots, 5$. For the boundary $y = 0$ we differentiate (3.9) with respect to each of the parameters to obtain,

$$\left(\frac{\partial}{\partial t} - rx \frac{\partial}{\partial x} - \kappa \theta \frac{\partial}{\partial y} + r \right) D_i = F_i(x, y, t; \mathbf{p}),$$

where

$$F_1 = \theta \frac{\partial g}{\partial y}, \quad F_2 = \kappa \frac{\partial g}{\partial y}$$

and $F_i = 0$ for $i = 3, 4, 5$. We do not have a sensitivity equation corresponding to the data r because r is a known deterministic constant for our purpose and which we do not estimate from the asset price data.

6 Numerical scheme for option price and derivatives under stochastic volatility

We use standard schemes for discretization of the partial differential equations for which stability of option price solutions has been shown. We apply a uniform space-time finite difference grid on the computational domain U_T . Let the number of grid steps be m, n and s in the x, y and t directions respectively. The grid steps in each direction are denoted

$$\Delta x = \frac{X_{max}}{m}; \quad \Delta y = \frac{Y_{max}}{n}; \quad \Delta t = \frac{T}{s}$$

We use superscripts to denote the time variable and subscripts to denote the spatial variable at the grid point values,

$$g_{ij}^k = g(x_i, y_j, t_k) = g(i\Delta x, j\Delta y, k\Delta t)$$

where $i = 0, 1, 2, \dots, m$; $j = 0, 1, 2, \dots, n$ and $k = 0, 1, 2, \dots, s$.

We apply the space discretization scheme used in [21] for American options to our European option pricing. We use a second order accurate finite difference scheme for the space derivatives. We use the classical central difference scheme for the first order derivatives and the usual three point scheme for the second order

derivatives. At the boundary $j = 0$ an upwind discretization scheme is used [16] for the derivative in the y direction in (3.9) which reads,

$$\delta_y g_{i0}^k = \frac{-3g_{i0}^k + 4g_{i1}^k - g_{i2}^k}{2\Delta y}.$$

The mixed derivatives are discretized using the seven point stencil described in [21]. The properties of the resulting discretization matrix \mathbf{A} are studied in [21]. The strictly diagonally dominant matrix with positive diagonal elements and non positive off diagonal elements is known to have good stability properties. In general the matrix \mathbf{A} is not an M-matrix but as remarked in [21] with sufficiently small time steps the resulting matrix is diagonally dominant. The space discretization leads to a semi discrete equation,

$$\frac{d\mathbf{g}}{dt} + \mathbf{A} = \mathbf{b}$$

where \mathbf{A} is an $mn \times mn$ matrix and \mathbf{b} is a column vector of length mn . The vector \mathbf{g} of length mn by mn is the option price at the grid points. The vector \mathbf{b} consists of terms due to the Neumann boundary condition in the x direction and does not depend on t .

For the time discretization we consider similarly to [24] the so called *Backward Difference Formula*, BDF2 scheme. This is an implicit scheme with second order accuracy. The stability of time discretization schemes is considered in [21] and [24] among others. At time $k\Delta t$ the BDF2 scheme reads,

$$\frac{3\mathbf{g}^{k+1} - 4\mathbf{g}^k + \mathbf{g}^{k-1}}{2\Delta t} + \mathbf{A}\mathbf{g}^{k+1} = \mathbf{b}$$

for $k = 1, 2, \dots, l-1$. This scheme is L -stable which is a stronger property than unconditional stability. Due to this property the BDF2 scheme does not cause oscillations in the solution. The other favorable properties of this scheme are considered in [24]. At each iterate of the BDF2 scheme we require the value of the last two iterates. As is typical we obtain the first iterate using an Implicit Euler scheme. That is, given \mathbf{g}^0 (the initial value), we obtain \mathbf{g}^1 using an implicit Euler scheme,

$$\frac{\mathbf{g}^1 - \mathbf{g}^0}{\Delta t} + \mathbf{A}\mathbf{g}^1 = \mathbf{b}$$

We have verified that this choice of space-time discretization of the initial/boundary-value problem gives us stable solutions for the option price at moderate grid sizes. At each time step we solve the following system of linear equation,

$$(6.17) \quad \left(\mathbf{I} + \frac{2}{3}\Delta t \mathbf{A} \right) \mathbf{g}^{k+1} = \frac{4}{3}\mathbf{g}^k - \frac{1}{3}\mathbf{g}^{k-1} + \Delta t \mathbf{b},$$

where \mathbf{I} is an mn by mn identity matrix. We solve this system of equations using an LU decomposition.

After we obtain the option price at the discrete grid points, we solve the sensitivity equations. The right hand side of the sensitivity equations is approximated using the central difference scheme for the first derivatives, the 3-point stencil for the second derivatives and the 7 point stencil for the mixed derivatives. We solve the sensitivity equations on the same grid that we used for the option price and use the same space-time discretization to discretize the sensitivity equations. We verify empirically that the scheme for the solution of the sensitivity equations converges.

7 Sensitivity to parameter estimation errors

We want to compute the sensitivity of S&P 500 options pricing to estimation errors, and as described earlier we use the VIX index as a proxy for the volatility value. The unknown parameter vector P of the underlying joint SDEs Heston model best fitted to 252 observed daily values of SPX and VPX were explicitly estimated in section(2) above by a vector Q , and the corresponding covariance matrix of the parameter estimation errors was also computed.

7.1 Computational Domain Numerically we solve for the option price viewed at each time t as a function $h(lsp, y, t)$ of t , of the log of the stock price ($lsp = \ln(x)$), and of the squared volatility y of the stock price x . This change of variable gives a transformed initial/boundary value problem that we solve on the following computational domain for $[lsp, y, t]$.

$$[lsp, y, t] \in (7, 8) \times (0, 1) \times (0, 0.25)$$

This domain was chosen after verifying carefully the robustness of the solution when the domain boundary $lsp = 7$ is pushed back to $lsp = 5$. Since the time increment between two consecutive daily observations is by convention set to $1/252$, the maturity date $T = .25$ indicates that we study an option with three months to maturity. We set the risk free rate of return at value $r = .01$ and we have estimated the market price of volatility risk by $\lambda = 2$. We use the following grid size for our computation,

$$\Delta lsp = .016, \quad \Delta y = .006, \quad \Delta t = .004$$

where Δlsp is the grid size for the space variable lsp computed as the “lsp” interval length (equal to 1 here) divided by the number of grid meshes in the lsp direction. Table(2) displays the CPU-times in seconds required for solving the partial differential equations for the option price PDE and all 5 sensitivity PDEs.

Grid size ($m \times n \times s$)	Δl	Δy	Δt	CPU-time (secs)
226800	1/60	1/60	1/252	95
308700	1/70	1/70	1/252	240

Table 2: Computing time for the option price and its derivatives

The LU decomposition is performed only once for each solution. The key computational cost is the LU decomposition and backward substitution for a matrix of size $mn \times mn$ where m and n are the grid sizes in the lsp and the y direction. The CPU-times increase more rapidly as we increase the spatial grid size mn than when we increase the grid size in time. For coarser grids the CPU-times are much lower but the solution is not accurate. We performed extensive numerical simulations to verify that the solutions converge and that the boundary conditions are satisfied.

7.2 The ranges of (SPX,VIX) data The Heston model parameters in section(2) were estimated using 252 daily (SPX,VIX) data for the year 2006. To present our sensitivity results within a realistic range, we consider 63 SPX and VIX index daily values for the first quarter of 2007, and use these 63 values as the successive values of asset price X_t and volatility (V_t), with the time “ t ” varying between 0 and $T = 0.25$ by time steps equal to $1/252$. In this period, the median SPX index value is 1426, and the median VIX index value is 11%.

We have computed precise sensitivity values for an option with strike price 1430 and three months to maturity ($t = 63/252$). The SPX index values during this period are between 1370 and 1460. The corresponding VIX values are between 10 % and 20 %. We display the dependence of option price sensitivities on asset price and volatility within these two realistic ranges. When we display the dependence of sensitivities on asset price, we keep the volatility fixed at 11% . To display sensitivities dependence on volatility, we keep the asset price fixed at 1408.

7.3 Impact of parameter estimation errors on option pricing We compute the estimation error on the option price for the estimated model $Q = (\hat{\kappa}, \hat{\theta}, \hat{\gamma}, \hat{\rho})$ given in Table(1). As described in section (4) we compute the impact of parameter estimation errors on the option price through the formula,

$$\varepsilon(Q) = \max_{P \in B(Q)} \delta(P)$$

were the L^2 -error $\delta(P)$ is computed by (4.11), and where the neighborhood $B(Q)$ is the product of the 4 intervals $B_i = [Q_i - \hat{\sigma}_i, Q_i + \hat{\sigma}_i]$ where $\hat{\sigma}_1 = \hat{\sigma}_\kappa, \hat{\sigma}_2 = \hat{\sigma}_\theta, \hat{\sigma}_3 = \hat{\sigma}_\gamma, \hat{\sigma}_4 = \hat{\sigma}_\rho$ are the estimates for standard deviation of κ, θ, γ and ρ displayed in Table(1).

We discretize the set $B(Q)$ using three grid points for each parameter, which gives us a grid of $3^4 = 81$ vectors. At each one of these 81 grid vectors P we compute the 4 partial derivatives of the option price with respect to the 4 key model parameters and the standard deviations of the 4 key parameter estimators, in order to compute the L^2 -error $\delta(P)$. We then take the maximum of these 81 values of $\delta(P)$ to obtain the option price error $\varepsilon(Q)$. In Fig.3 we illustrate the impact of parameter estimation errors on the pricing of 2 options (written on the SPX index), by plotting the computed option price $g(Q)$ at Q , as a function of either the asset price or the volatility, and on two associated dotted curves, we display $g(Q) - \varepsilon(Q)$ and $g(Q) + \varepsilon(Q)$.

7.4 Sensitivity of option prices to the parameters of the joint SDEs Heston model The sensitivity of the option price to each model parameter was defined in section(4). In Fig.4, for each model parameter, we show how the corresponding option price sensitivity depends on asset price (for volatility fixed at 11%) and on volatility (for asset price fixed at 1408). The model parameters vector PAR is here fixed at the value $PAR = Q$ estimated from the 2006 (SPX,VIX) dataset.

For each parameter PAR_i , we also compute the empirical mean sensitivity $MSEN_{PAR_i}$ of the option price with respect to PAR_i . This mean sensitivity computation is based on the empirical distribution of the asset price over the life of the option. The mean sensitivity of the option price studied here are given by $MSEN_\kappa = 0.42, MSEN_\theta = 1.57, MSEN_\gamma = 0.02, MSEN_\rho = 0.04$. The mean option price during this period is 30.37 and the median option price is 27.60. The option price is most sensitive to errors on the parameter θ .

7.5 Derivative of options with respect to market price λ of volatility risk We compute the derivative $D_5(g(Q)) = \partial_\lambda g(Q)$ of the option price with respect to the market price of risk, to evaluate the option price sensitivity to errors on this unknown market price λ . Table(3) presents $|\partial_\lambda g(Q)|$ at realistic levels of asset price and volatility values. These derivatives are computed at the estimated model Q for an option with strike price equal to 1430 and 3 months to maturity.

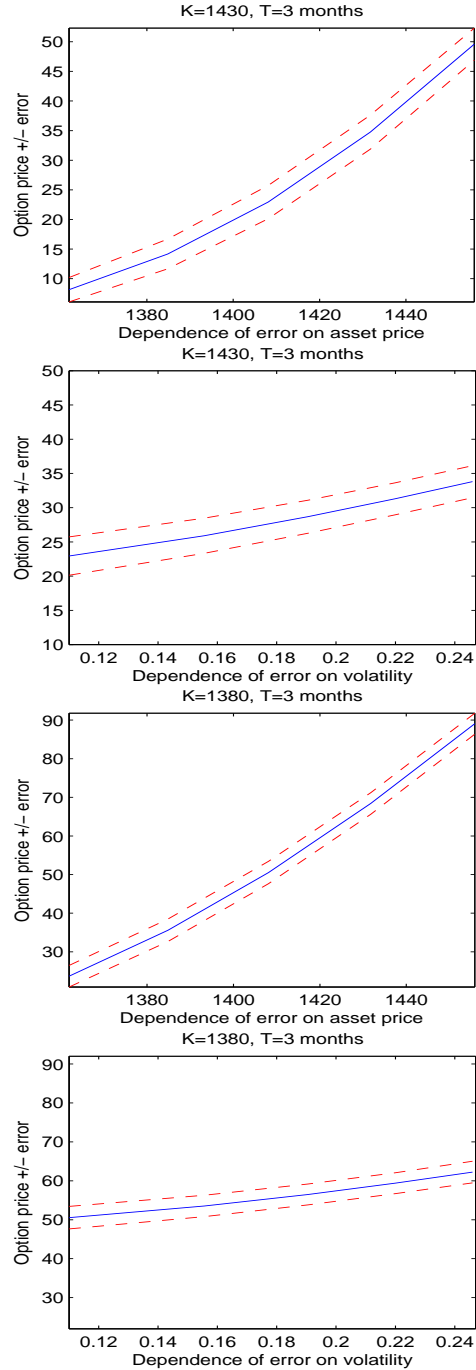


Figure 3: Blue curve = computed option price for the estimated S&P 500 model. Dotted curves display option price plus or minus computed error margin $\varepsilon(Q)$

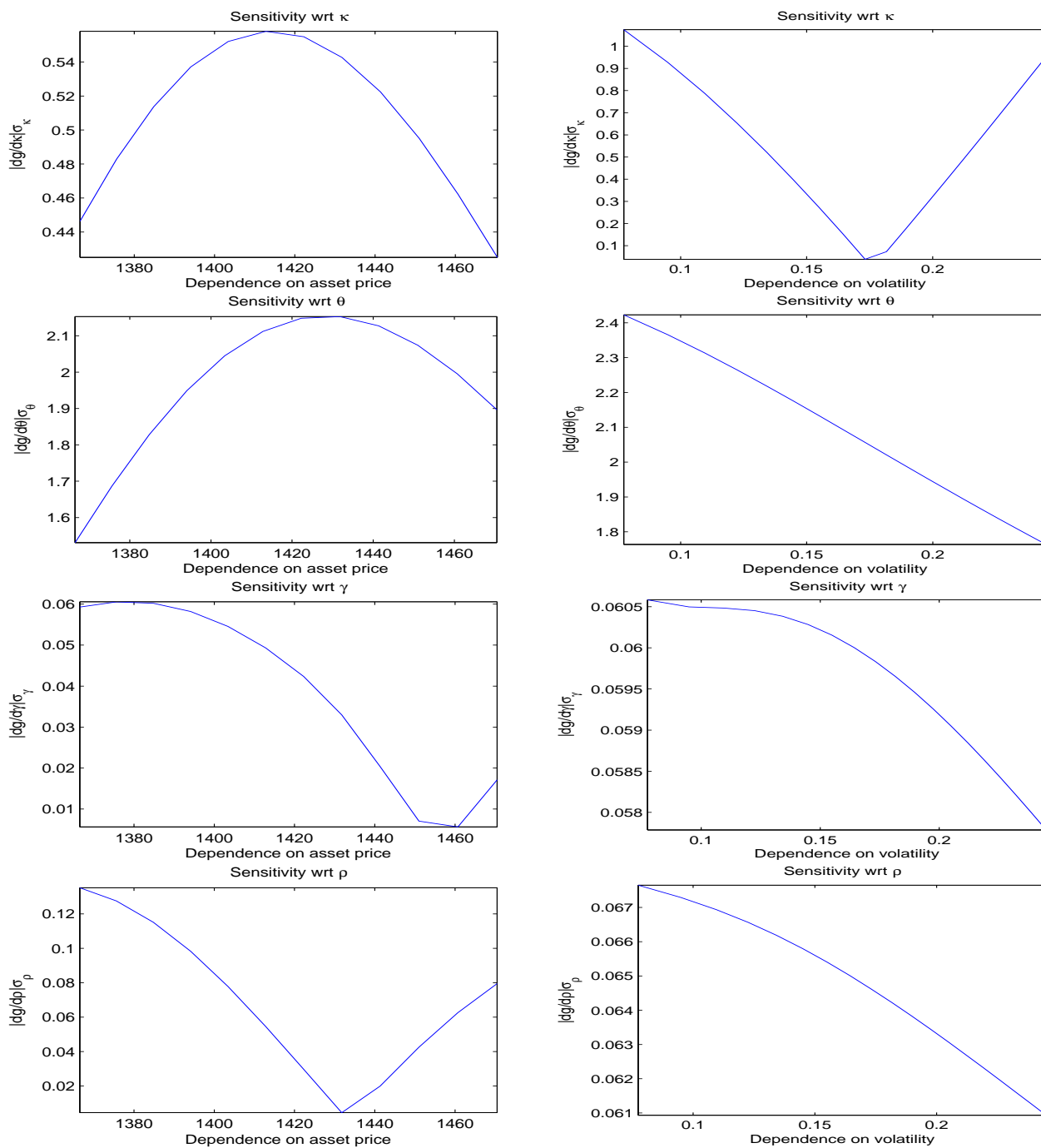


Figure 4: The graphs in the left column display option price sensitivity to κ , θ , γ , ρ resp., as functions of asset prices for fixed volatility 11%. The analogous graphs in the right column display how these sensitivities depend on volatility for fixed asset price 1408. These results correspond to an European option with strike =1430 and 3 months to maturity

Asset price	Volatility	Derivative	Option price
1422	11%	1.55	29.90
1422	19%	1.64	35.74
1357	11%	.96	5.65
1357	19%	1.07	9.29
1480	11%	1.30	67.43
1480	19%	1.45	72.35

Table 3: absolute value of derivative of the option price with respect to λ computed at $\lambda = 2$

8 Conclusion

We have defined a generic mathematical approach, and implemented an innovative set of corresponding numerical algorithms to compute option price sensitivities to estimation errors on each model parameter of the joint SDEs Heston model for the underlying asset price and volatility.

Our approach combines efficient estimation of the underlying model parameters and of their variances and covariances, with the numerical solution of 6 parabolic equations on R^2 , satisfied by the option price and by its partial derivatives with respect to the model parameters of the underlying joint SDEs model.

We also had to develop and apply a consistent method to estimate the market price of the volatility risk on concrete option data.

We have applied our approach to model the 2006 S&P 500 daily data by a Heston pair of coupled SDEs, and to study the option price sensitivities of several European call options based on this index. The results are coherent and quite satisfactory from a pragmatic point of view.

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Mathematical and Computer Modelling, **47**, 2008, 422–430

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