Journal of Mathematical Sciences, Vol. 156, No. 5, 2009

ON ANALYTIC SOLUTIONS OF THE HEAT EQUATION WITH AN OPERATOR COEFFICIENT

A. Vershynina* **and S. Gefter*** UDC 517.968; 517.983

Let A be a bounded linear operator on a Banach space and let g a be vector-valued function that is analytic in a neighborhood of the origin of $\mathbb{R}$. We obtain conditions of the existence of analytic solutions for the Cauchy problem $\begin{cases} \frac{\partial u}{\partial t} = A^2 \frac{\partial^2 u}{\partial x^2}, \\ u(0,x) = g(x). \end{cases}$ *Moreover, we consider a representation of the solution of this problem as a Poisson integral and study the Cauchy problem for the corresponding inhomogeneous equation. Bibliography: 22 titles.*

1. Introduction

The Cauchy theorem on analytic solutions of differential equations with analytic coefficients is well known in the theory of ordinary differential equations (see, e.g., [1]). For the class of so-called normal partial differential equations, a similar theorem had been proved by Cauchy and Kovalevskaya [2–4]. Moreover, Kovalevskaya [5] showed that if an equation is not normal, then a Cauchy problem for this equation may fail to have analytic solutions. Let us consider the famous example of Kovalevskaya:

$$\begin{cases} \dfrac{\partial u}{\partial t} = a^2 \dfrac{\partial^2 u}{\partial x^2}, \\ u(0,x) = \dfrac{b}{1-x} \end{cases} \tag{1}$$

(in what follows, it is convenient for us to consider an arbitrary coefficient b in the initial condition).

It is easy to check that the following power series:

$$\sum_{n,m=0}^{\infty} \frac{(m+2n)!}{m!n!} a^{2n} b t^n x^m \tag{2}$$

is a *formal* solution of problem (1). Therefore, for $a \neq 0$ and $b \neq 0$, the Cauchy problem (1) does not have solutions that are *analytic* in a neighborhood of zero. The research initiated by Kovalevskaya was continued in numerous papers (see, e.g., [6–18]).

In this paper, we consider the following *operator analog* of the Cauchy problem (1):

$$\begin{cases} \dfrac{\partial u}{\partial t} = A^2 \dfrac{\partial^2 u}{\partial x^2}, \\ u(0,x) = \dfrac{b}{1-x}, \end{cases} \tag{3}$$

and a more general Cauchy problem:

$$\begin{cases} \dfrac{\partial u}{\partial t} = A^2 \dfrac{\partial^2 u}{\partial x^2}, \\ u(0,x) = g(x). \end{cases} \tag{4}$$

Here A is a bounded linear operator on a Banach space E, $b \in E$, and g is a vector-valued function that is analytic in a neighborhood of the origin. By formal analogy with the equation in (1), Eq. (4) is also called "the heat equation." Note that in some interesting examples, our "heat equation" is a hyperbolic partial differential equation (see Remark 3.7). We consider solutions of the Cauchy problem (4) that are analytic in a neighborhood of the origin of $\mathbb{R}\times\mathbb{R}$. By a solution of problem (4) we mean a *local* analytic solution, i.e., a vector-valued function of real variables t and x that is analytic in a neighborhood of zero, satisfies the equation in this neighborhood,

*V. N. Karazin Kharkiv National University, Kharkiv, Ukraine, e-mail: gefter@univer.kharkov.ua, avershynina@yahoo.com.

Translated from *Zapiski Nauchnykh Seminarov POMI*, Vol. 355, 2008, pp. 139–162. Original article submitted June 2, 2008.

1072-3374/09/1565-0799

and satisfies the initial condition in a neighborhood of the origin of $\mathbb{R}$. The *formal* solution of Cauchy problem (3) looks like the scalar one:

$$\sum_{n,m=0}^{\infty} \frac{(m+2n)!}{m!n!} A^{2n} b t^n x^m. \tag{5}$$

It is obvious that if $A = 0$, then the function $u(t,x) = \sum_{n=0}^{\infty} bx^n = \frac{b}{1-x}$ is an analytic solution of problem (3). Similarly, there exists an analytic solution of problem (3) if A is a *nilpotent* operator, i.e., $A^k = 0$ for some k. We also consider more general operators that are close to zero in the spectral sense (namely, *quasinilpotent* ones). Recall that an operator A is called quasinilpotent if the spectrum $\sigma(A)$ of A consists of the single point $\lambda = 0$. We show that series (5) can be convergent in a neighborhood of the origin if A is a quasinilpotent operator satisfying some additional assumption (see Proposition 3.2).

Now we formulate the main result of our paper.

Theorem 3.8. *The following conditions are equivalent:*

(1) *The Cauchy problem* (3) *has an analytic solution for each vector* $b \in E$*;*

(2) *the Cauchy problem* (4) *has an analytic solution for any vector-valued function* $g(x)$ *that is analytic in a neighborhood of zero;*

(3) *the operator* A *is quasinilpotent, and the Fredholm resolvent* $F_{A^2}(z) = (1 - zA^2)^{-1}$ *of the operator* A^2 *is an entire function of exponential type (i.e.,* $\|F_{A^2}(z)\| \le Ce^{\beta|z|}$ *for some constants* C *and* β*).*

Moreover, if these conditions are fulfilled, then the solution of the Cauchy problem (4) *is unique and has the following explicit form:*

$$u(t,x) = g(x) + \sum_{n=1}^{\infty} \frac{t^n}{n!} A^{2n} g^{(2n)}(x)$$

(see Remark 5.2*).*

Thus, if the Cauchy problem (4) has an analytic solution for any analytic initial condition, then the operator A is close to zero in the spectral sense. In particular, in the finite-dimensional case, the equation from the Cauchy problem (4) is of the form

$$\frac{\partial u_k}{\partial t} = \sum_{j=1}^{m} c_{kj} \frac{\partial^2 u_j}{\partial x^2}, \quad k = 1, \dots, m \quad (m = \dim E),$$

where the matrix $C = (c_{kj})$ is nilpotent (see Corollary 3.4). Certainly, in the given particular case, this fact is a simple corollary of the general theorem obtained by Mizohata (see [10, Sec. 3, Theorem 1]).

Theorem 3.8 can be considered as one more illustration of unusual properties of objects connected with quasinilpotent operators (see, for example, [19, Secs. 4.6 and 4.10] and [22]).

Some examples of explicit solutions of the Cauchy problem (3) are given in Sec. 3 (see Corollaries 3.3–3.5, Example 3.6, and Remark 3.7).

Our study of the Cauchy problem (4) is based on the concept of A-holomorphic formal power series (see Definition 2.1), which was investigated in the paper [22]. In [22], this concept was used in the study of holomorphic solutions of the equation $z^2Aw' + g(z) = w$, where A is a quasinilpotent linear operator on a Banach space. Let us note that each condition of the Theorem 3.8 is equivalent to the A^2-holomorphicity of the formal power series $\psi(\zeta) = \sum_{n=0}^{\infty} \frac{(2n)!}{n!} \zeta^n$ (see Proposition 2.9). The concept of A-holomorphicity is considered in Sec. 2.

In Sec. 4, we consider a representation of the solution of the Cauchy problem (4) as a Poisson integral. In our situation (i.e., if A is quasinilpotent), the operator analog of the heat kernel $H_A(t,\xi) = \frac{1}{2A\sqrt{\pi t}} \exp\{-\frac{\xi^2}{4A^2 t}\}$ certainly has no usual sense. We consider H_A as a vector-valued distribution, where the space of "test functions" is the space of all convergent power series with coefficients from E (see Definition 4.1 and Proposition 4.3). Then we show that the solution of the Cauchy problem (4) can be represented as the convolution of H_A with the initial condition g (see Theorem 5.1).

In Sec. 5, we study the Cauchy problem for the inhomogeneous equation:

$$\begin{cases} \dfrac{\partial u}{\partial t} = A^2 \dfrac{\partial^2 u}{\partial x^2} + f(t,x), \\ u(0,x) = 0. \end{cases}$$

We show that if the conditions of Theorem 3.8 are fulfilled, then the analytic solution of this Cauchy problem can be found as a series with respect to the "small parameter" A (see Theorem 5.1):

$$u(t,x) = \sum_{k=0}^{\infty} A^{2k} u_k(t,x).$$

2. Preliminaries

Let E be a complex Banach space, let $A : E \to E$ be a bounded linear operator, let $b \in E$, and let $f(\zeta) = \sum_{n=0}^{\infty} c_n \zeta^n$ be a formal power series with coefficients from $\mathbb{C}$. Define

$$f(zA) = \sum_{n=0}^{\infty} c_n A^n z^n, \quad z \in \mathbb{C}, \tag{6}$$

and

$$f(zA)b = \sum_{n=0}^{\infty} c_n A^n b z^n, \quad z \in \mathbb{C}. \tag{7}$$

Then $f(zA)$ is a power series with coefficients from the algebra $B(E)$ of all bounded operators in the space E, and $f(zA)b$ is a power series with coefficients from E. The radius of convergence of series (6) is denoted by $R_A(f)$, and that of series (7) is denoted by $R_{A,b}(f)$.

Definition 2.1. *The power series $f(\zeta)$ is called A-holomorphic if $R_A(f) > 0$ and (A,b)-holomorphic if $R_{A,b}(f) > 0$.*

It is obvious that an A-holomorphic power series is (A,b)-holomorphic for all vectors $b \in E$, and $R_{A,b}(f) \geq R_A(f)$. Moreover, if $|z| < R_A(f)$, then the sum of the series in the right-hand side of equality (7) is the result of the action of the operator $f(zA)$ on b.

Remark 2.2. Assume that the power series f has a positive radius of convergence $R(f)$. Then this series is A-holomorphic for each bounded operator A. Moreover, if $\rho(A)$ is the spectral radius of the operator A and $|z|\rho(A) < R(f)$, then $f(zA)$ is well defined as the action of the holomorphic function f on the operator zA.

Example 2.3. Assume that $b \in \ker A^m$ for some $m \in \mathbb{N}$. Then

$$f(zA)b = \sum_{n=0}^{m-1} c_n A^n b z^n,$$

i.e., every power series $f(\zeta)$ is (A,b)-holomorphic.

If the space E is finite-dimensional, then the converse statement also holds in the situation which is most interesting for us (see [22, Proposition 1.4]).

Proposition 2.4. *Let $\dim E < \infty$, let f be a power series, and let the radius of convergence of f be 0. If f is (A,b)-holomorphic, then $b \in \ker A^m$ for some $m \in \mathbb{N}$.*

In the Hilbert space case, the following analog of Proposition 2.4 is true.

Proposition 2.5. *Let E be a Hilbert space, let $f(\zeta) = \sum_{n=0}^{\infty} c_n \zeta^n$ be a power series, and let the radius of convergence of f be equal to zero. If A is a normal bounded operator and f is (A,b)-holomorphic, then $b \in \ker A$.*

Proof. Let $R_{A,b}(f) > 0$. According to the vector analog of the Cauchy–Hadamard formula,

$$\frac{1}{R_{A,b}(f)} = \overline{\lim_{n\to\infty}} \sqrt[n]{|c_n| \|A^n b\|} < \infty.$$

Let us show that there exists $\lim\limits_{n\to\infty} \sqrt[n]{\|A^n b\|}$. According to the spectral theorem, we can identify E with $L^2(X,\mu)$ for some measure space (X,μ) and consider A as the multiplication operator:

$$(Ab)(x) = a(x)b(x), \quad \text{where} \quad a \in L^\infty(X,\mu).$$

Then $\sqrt[n]{\|A^n b\|} = \sqrt[2n]{\int\limits_X |a(x)|^{2n}|b(x)|^2 d\mu}$, and this sequence converges to the norm of $a(x)$ in the space $L^\infty(X, \mu_b)$, where $d\mu_b = |b(x)|^2 d\mu$. On the other hand, $\overline{\lim\limits_{n\to\infty}} \sqrt[n]{|c_n|} = \infty$. Hence, $\lim\limits_{n\to\infty} \sqrt[n]{\|A^n b\|} = 0$. Therefore, $a(x) = 0$ μ_b-almost everywhere, i.e., $a(x)b(x) = 0$ μ-almost everywhere. Thus, $Ab = 0$. □

Using Example 2.3, it is easy to find examples of (A, b)-holomorphic formal power series that are not A-holomorphic. However, using the Baire category theorem, one can prove the following statement (see [22, Theorem 1.5]).

Theorem 2.6. *If the formal power series f is (A, b)-holomorphic for all $b \in E$, then it is A-holomorphic.*

The following statement shows that if A is not quasinilpotent, then the concept of A-holomorphic formal power series coincides with the usual concept of holomorphic power series (see [22, Proposition 1.7]).

Proposition 2.7. *If the operator A has a positive spectral radius, then a power series f is A-holomorphic if and only if it has a positive radius of convergence. Thus, if f has zero radius of convergence and f is A-holomorphic, then A is quasinilpotent.*

The following two formal power series play an important role in our further study:

$$\varphi(\zeta) = \sum_{n=0}^{\infty} n!\zeta^n \quad \text{and} \quad \psi(\zeta) = \sum_{n=0}^{\infty} \frac{(2n)!}{n!}\zeta^n.$$

Lemma 2.8. *Let $A : E \to E$ be a bounded linear operator and let $b \in E$. Then*
(1) *ψ is (A, b)-holomorphic if and only if φ is (A, b)-holomorphic;*
(2) *ψ is A-holomorphic if and only if φ is A-holomorphic.*

Proof. It is enough to notice that, according to the Stirling formula,

$$\sqrt[n]{\frac{(2n)!}{n!}} \sim \frac{4n}{e} \quad \text{and} \quad \sqrt[n]{n!} \sim \frac{n}{e}. \qquad \square$$

Recall that an entire function $g(z)$ with values in a Banach space is called a function of exponential type if $\|g(z)\| \le Ce^{\beta|z|}$ for some constants C and β. Recall also that a bounded linear operator A is quasinilpotent if and only if its Fredholm resolvent $(1 - zA)^{-1}$ is an entire function (see [19, Chap. 4]).

Proposition. *The Fredholm resolvent $(1 - zA)^{-1}$ of the operator A is an entire function of exponential type if and only if the series $\psi(\zeta) = \sum\limits_{n=0}^{\infty} \frac{(2n)!}{n!}\zeta^n$ is A-holomorphic.*

Proof. It follows from Lemma 2.8 that we can consider the power series $\varphi(\zeta) = \sum\limits_{n=0}^{\infty} n!\zeta^n$ instead of $\psi(\zeta)$. Let φ be A-holomorphic. Then

$$\frac{1}{R_A(\varphi)} = \overline{\lim_{n\to\infty}} \sqrt[n]{n!\|A^n\|} < +\infty, \tag{8}$$

and the operator A is quasinilpotent. Now identity (8) is equivalent to the statement that the entire function $(1 - zA)^{-1} = \sum\limits_{n=0}^{\infty} A^n z^n$ is of exponential type (see [21, p. 95]). □

Let us give an example where the function $\psi(zA)$ can be computed explicitly.

Example 2.10. Let $E = C[0, 1]$ and let A_1 be the integration operator:

$$(A_1 b)(s) = \int_0^s b(y)dy, \quad b \in E.$$

It is well known that

$$(A_1^n b)(s) = \frac{1}{(n-1)!} \int_0^s (s - y)^{n-1} b(y)dy, \tag{9}$$

$\|A_1^n\| \le \dfrac{1}{n!}$, and $A_1 = A^2$, where

$$(Ab)(s) = \frac{1}{\sqrt{\pi}} \int\limits_0^s \frac{b(y)}{\sqrt{s-y}} dy.$$

It is easy to check that the series ψ is A_1-holomorphic and $R_{A_1}(\psi) = 1/4$, i.e., ψ is A^2-holomorphic and $R_{A^2}(\psi) = 1/4$. For the explicit calculation of the operator $\psi(zA^2)$, note that

$$1 + \sum_{n=1}^{\infty} \frac{(2n)!}{(n!)^2} (\frac{x}{2})^{2n} = \frac{1}{\sqrt{1-x^2}}, \quad |x| < 1.$$

Hence,

$$\sum_{n=1}^{\infty} \frac{(2n)!}{n!(n-1)!} (\frac{x}{2})^{2n-2} = \frac{2}{(1-x^2)^{3/2}}, \quad |x| < 1,$$

and

$$\sum_{n=1}^{\infty} \frac{(2n)!}{n!(n-1)!} \gamma^{n-1} = \frac{2}{(1-4\gamma)^{3/2}}, \quad |\gamma| < \frac{1}{4}. \tag{10}$$

Now from (9) and (10) we deduce that

$$(\psi(zA^2)b)(s) = b(s) + 2z \int\limits_0^s \frac{b(y)dy}{(1-4z(s-y))^{3/2}}, \quad |z| < 1/4. \tag{11}$$

3. Main result

Let E be a Banach space, let $A : E \to E$ a bounded linear operator, and let g be a E-valued function that is analytic in a neighborhood of zero. In this section, we consider the problem of solution existence for the Cauchy problem

$$\begin{cases} \dfrac{\partial u}{\partial t} = A^2 \dfrac{\partial^2 u}{\partial x^2}, \\ u(0,x) = g(x). \end{cases} \tag{12}$$

By a solution of problem (12) we mean a *local analytic solution*, i.e., a vector-valued function of real variables t and x that is analytic in a neighborhood of zero, satisfies the equation in this neighborhood, and the initial condition in a neighborhood of the point $x_0 = 0$.

At first, consider only an algebraic situation in which g is a formal power series.

Lemma 3.1. *Let $g(x) = \sum\limits_{m=0}^{\infty} b_m x^m$ be a formal power series with coefficients from E. Then the Cauchy problem* (12) *has a unique formal solution*

$$u(t,x) = \sum_{n,m=0}^{\infty} \frac{(m+2n)!}{m!n!} A^{2n} b_{m+2n} t^n x^m = \sum_{n=0}^{\infty} \frac{t^n}{n!} A^{2n} g^{(2n)}(x).$$

Proof. Assume that

$$u(t,x) = \sum_{n,m=0}^{\infty} c_{nm} t^n x^m,$$

$c_{nm} \in E$, is a formal solution of the Cauchy problem (12). After a formal substitution into the equation, we see that

$$\sum_{n,m=0}^{\infty} (n+1) c_{n+1m} t^n x^m = \sum_{n,m=0}^{\infty} (m+2)(m+1) A^2 c_{nm+2} t^n x^m$$

and

$$\sum_{m=0}^{\infty} c_{0m} x^m = \sum_{m=0}^{\infty} b_m x^m.$$

Hence,

$$(n+1)c_{n+1m} = (m+2)(m+1)A^2 c_{nm+2}$$

and

$$c_{0m} = b_m, \quad n, m \geq 0,$$

i.e.,

$$c_{n,m} = \frac{(m+2n)!}{m!n!} A^{2n} b_{m+2n}, \quad n, m \geq 0.$$

Thus,

$$u(t,x) = \sum_{n,m=0}^{\infty} \frac{(m+2n)!}{m!n!} A^{2n} b_{m+2n} t^n x^m$$

is the unique formal solution.

It is easy to check that this solution can be represented as $u(t,x) = \sum\limits_{n=0}^{\infty} \frac{t^n}{n!} A^{2n} g^{(2n)}(x)$, and the expression on the right-hand side is a well-defined formal power series in variables t, x. The lemma is proved. $\square$

Now for $b \in E$ we consider the following special Cauchy problem:

$$\begin{cases} \dfrac{\partial u}{\partial t} = A^2 \dfrac{\partial^2 u}{\partial x^2}, \\ u(0,x) = \dfrac{b}{1-x}. \end{cases} \tag{13}$$

Proposition 3.2. *The Cauchy problem* (13) *has a solution if and only if the formal power series* $\psi(\zeta) = \sum\limits_{n=0}^{\infty} \frac{(2n)!}{n!} \zeta^n$ *is* (A^2, b)*-holomorphic (see Definition* 2.1*). Moreover, the solution is unique, and it can be represented by the following two series:*

$$u(t,x) = \sum_{n,m=0}^{\infty} \frac{(m+2n)!}{m!n!} A^{2n} b t^n x^m$$

and

$$u(t,x) = \psi\Big(\frac{tA^2}{(1-x)^2}\Big) \frac{b}{1-x} = \sum_{n=0}^{\infty} \frac{(2n)!}{n!} \frac{A^{2n} b t^n}{(1-x)^{2n+1}},$$

$|t| < T$, $|x| < R$, *where* $R \in (0,1)$ *and* $T = (1-R)^2 R_{A^2,b}(\psi)$.

Proof. According to Lemma 3.1, the unique formal solution of the Cauchy problem (13) is

$$u(t,x) = \sum_{n,m=0}^{\infty} \frac{(m+2n)!}{m!n!} A^{2n} b t^n x^m.$$

Assume that this series converges for $|t| < T$ and $|x| < R$, where $R \in (0,1)$. Since

$$\sum_{m=0}^{\infty} \frac{(m+2n)!}{m!n!} x^m = \frac{1}{(1-x)^{2n+1}},$$

$$u(t,x) = \sum_{n=0}^{\infty} \frac{(2n)!}{n!} \Big(\sum_{m=0}^{\infty} \frac{(m+2n)!}{m!n!} x^m \Big) A^{2n} b t^n = \sum_{n=0}^{\infty} \frac{(2n)!}{n!} \frac{1}{(1-x)^{2n+1}} A^{2n} b t^n.$$

Therefore,

$$\overline{\lim_{n\to\infty}} \sqrt[n]{\frac{(2n)!}{n!} \frac{1}{(1-x)^{2n+1}} \|A^{2n} b\|} < +\infty,$$

i.e., ψ is (A^2, b)-holomorphic.

On the other hand, let ψ be (A^2, b)-holomorphic. Then the power series $\sum_{n=0}^{\infty} \frac{(2n)!}{n!} A^{2n} b \frac{t^n}{(1-x)^{2n+1}}$ converges if $\frac{|t|}{(1-x)^2} < R_{A^2,b}(\psi)$. Therefore, if $R \in (0,1)$, then the power series $u(t,x) = \sum_{n,m=0}^{\infty} \frac{(m+2n)!}{m!n!} A^{2n} b t^n x^m$ converges for $|x| < R$, $|t| < (1-R)^2 R_{A^2,b}(\psi)$, and

$$u(t,x) = \sum_{n=0}^{\infty} \frac{(2n)!}{n!} \frac{A^{2n} b t^n}{(1-x)^{2n+1}}, \quad |t| < T, \quad |x| < R. \qquad \square$$

According to 2.4 and 2.5, we deduce the following corollaries from Proposition 3.2.

Corollary 3.3. *Let* $\dim E < +\infty$. *Then the Cauchy problem* (13) *has a solution if and only if* $b \in \ker A^k$ *for some* $k \in \mathbb{N}$. *Moreover, if* $b \in \ker A^{2N+1}$, *then the solution of this problem is of the form*

$$u(t,x) = \sum_{n=0}^{N} \frac{(2n)!}{n!} \frac{A^{2n} b t^n}{(1-x)^{2n+1}}, \quad t \in \mathbb{R}, \quad |x| < 1.$$

Corollary 3.4. *Let* $\dim E < +\infty$. *Then the Cauchy problem* (13) *has a solution for each vector* $b \in E$ *if and only if the operator* A *is nilpotent.*

Corollary 3.5. *Let* E *be a Hilbert space, let* $b \in E$, *and let* A *be a bounded normal operator. Then the Cauchy problem* (13) *has a solution if and only if* $b \in \ker A$.

Now we give an example of a nontrivial explicit solution of Cauchy problem (13).

Example 3.6. Let $E = C[0,1]$ and let A be the square root from an integration operator:

$$(Ab)(s) = \frac{1}{\sqrt{\pi}} \int_0^s \frac{b(y)}{\sqrt{s-y}} dy.$$

Then ψ is A^2-holomorphic, and $R_{A^2,b}(\psi) = 1/4$ (see Example 2.10). According to Proposition 3.2, the solution of the Cauchy problem (13) is of the form

$$u(t,x) = \psi\Big(\frac{tA^2}{(1-x)^2}\Big) \frac{b}{1-x}, \quad x \in (-1,1), \quad |t| < \frac{1}{4}(1-x)^2.$$

Now equality (11) in Example 2.10 shows that

$$[u(t,x)](s) = \frac{b(s)}{1-x} + 2t \int_0^s \frac{b(y)dy}{((1-x)^2 - 4t(s-y))^{3/2}}, \tag{14}$$

$x \in (-1,1)$, $|t| < \frac{1}{4}(1-x)^2$, $s \in [0,1]$.

Remark 3.7. Example 3.6 shows that the series sum $\psi(zA^2) = \sum_{n=0}^{\infty} \frac{(2n)!}{n!} A^{2n} z^n$ presents implicitly in a formula for a solution of the 2-D wave equation with some special initial conditions. Indeed, in this example, the Cauchy problem (13) can be written in the following form:

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x,s) = \displaystyle\int_0^s \frac{\partial^2 u}{\partial x^2}(t,x,y)dy, \\ u(0,x,s) = \dfrac{b(s)}{1-x}, \quad s \in [0,1]. \end{cases}$$

Thus, the function $u(t,x,s)$ satisfies the partial differential equation:

$$\frac{\partial}{\partial s}\Big(\frac{\partial u}{\partial t}\Big)=\frac{\partial^2 u}{\partial x^2}$$

and the conditions $u(0,x,s)=\frac{b(s)}{1-x}$, $s\in[0,1]$, and $\frac{\partial u}{\partial t}(t,x,0)=0$.

Now from (14) we conclude that the function

$$u(t,x,s)=\frac{b(s)}{1-x}+2t\int\limits_0^s\frac{b(y)dy}{((1-x)^2-4t(s-y))^{3/2}},\quad x\in(-1,1),\quad |t|<\frac{1}{4}(1-x)^2,$$

$s\in[0,1]$, is a solution of Eq. (15). If $b\in C^1[0,1]$, then the function $u(t,x,s)$ is differentiable with respect to s. In this case, Eq. (15) may be rewritten in the usual form $\frac{\partial^2 u}{\partial t\partial s}=\frac{\partial^2 u}{\partial x^2}$. By a linear substitution, this equation can be reduced to the wave equation.

Finally, consider the Cauchy problem (12) with an arbitrary analytic vector-valued function g, where

$$g(x)=\sum_{m=0}^{\infty}b_m x^m,\quad |x|<R(g).$$

Theorem 3.8. *The following conditions are equivalent:*

(1) *The Cauchy problem* (13) *has an analytic solution for each vector* $b\in E$*;*

(2) *the Cauchy problem* (12) *has an analytic solution for each vector-valued function* $g(x)$ *that is analytic in a neighborhood of zero;*

(3) *the operator* A *is quasinilpotent (i.e., the spectrum of* A *contains of the point* 0 *only), and the Fredholm resolvent* $(1-zA^2)^{-1}$ *of the operator* A^2 *is an entire function of exponential type.*

Moreover, if at least one of these conditions is true, then the Cauchy problem (12) *has a unique analytic solution*

$$u(t,x)=\sum_{n,m=0}^{\infty}\frac{(m+2n)!}{m!n!}A^{2n}b_{m+2n}t^n x^m,$$

and this series converges for $|t|<T_0$, $|x|<R_0$, *where* $T_0=\alpha_0 R_{A^2}(\psi)R(g)^2$, $R_0=\beta_0\beta_1 R(g)$, *and* α_0, β_0, β_1 *are arbitrary constants satisfying the conditions* $\alpha_0,\beta_0,\beta_1\in(0,1)$ *and* $\alpha_0<\beta_1^2(1-\beta_0)^2$.

Proof. According to Proposition 3.2 and Theorem 2.6, condition (1) is equivalent to the fact that the power series $\psi(\zeta)=\sum\limits_{n=0}^{\infty}\frac{(2n)!}{n!}\zeta^n$ is A^2-holomorphic. Hence, conditions (1) and (3) are equivalent (see Lemma 2.8 and Proposition 2.9). It is obvious that (1) follows from (2). We claim that condition (2) follows from the A^2-holomorphicity of the power series $\psi(\zeta)$. According to Lemma 3.1,

$$u(t,x)=\sum_{n,m=0}^{\infty}\frac{(m+2n)}{m!n!}A^{2n}b_{m+2n}t^n x^m$$

is the unique formal solution of the Cauchy problem (12). Now we show that there exist positive T_0 and R_0 such that this series converges for $|t|<T_0, |x|<R_0$. Consider $c_1,c_2,c_3,c_4\in(0,1)$ with $c_3<c_1c_2^2(1-c_4)^2$ (for example, $c_1=9/10$, $c_2=3/4$, $c_3=1/8$, and $c_4=1/2$) and $r_1=c_1R_{A^2}(\psi)$. Then the series $\sum\limits_{n=0}^{\infty}\frac{(2n)!}{n!}\|A^{2n}\|r_1^n$ converges. Therefore, there exists a constant M_1 such that

$$\frac{(2n)!}{n!}\|A^{2n}\|\le\frac{M_1}{r_1^n},\quad n=0,1,\dots.$$

Let $r_2=c_2R(g)$. Then there exists a constant $M_2>0$ such that $\|b_m\|\le\frac{M_2}{r_2^m}$, $m=0,1,\dots$. Hence,

$$\frac{(2n)!}{n!}\sum_{m=0}^{\infty}\frac{(m+2n)!}{m!(2n)!}\|A^{2n}\|\|b_{m+2n}\||x|^m\le\frac{(2n)!}{n!}\frac{M_2\|A^{2n}\|}{r_2^{2n}}\sum_{m=0}^{\infty}\frac{(m+2n)!}{m!(2n)!}\Big(\frac{|x|}{r_2}\Big)^m$$

$$= \frac{(2n)!M_2\|A^{2n}\|}{n!r_2^{2n}} \frac{1}{(1-\frac{|x|}{r_2})^{2n+1}} \le \frac{M_1M_2}{r_1^n r_2^{2n}} \frac{1}{(1-\frac{|x|}{r_2})^{2n+1}}$$

for $|x| < r_2$ and $n = 0, 1, \dots$.

Now let $T_0 = c_3 R_{A^2}(\psi) R(g)^2$ and $R_0 = c_2 c_4 R(g)$. If $|t| < T_0$ and $|x| < R_0$, then

$$\frac{|t|}{r_1 r_2^2 (1-\frac{|x|}{r_2})^2} < \frac{c_3 R_{A^2}(\psi) R(g)^2}{c_1 R_{A^2}(\psi) c_2^2 R(g)^2 (1-\frac{R_0}{r_2})^2} = \frac{c_3}{c_1 c_2^2 (1-c_4)^2} < 1,$$

i.e., the series

$$\sum_{n=0}^{\infty} \frac{1}{r_1^n r_2^{2n}} \frac{|t|^n}{(1-\frac{|x|}{r_2})^{2n+1}}$$

and

$$\sum_{n=0}^{\infty} \frac{(2n)!}{n!} \bigg(\sum_{m=0}^{\infty} \frac{(m+2n)!}{m!(2n)!} \|A^{2n}\| \|b_{m+2n}\| |x|^m \bigg) |t|^n$$

converge. Thus, the series

$$\sum_{n,m=0}^{\infty} \frac{(m+2n)!}{m!n!} \|A^{2n}\| \|b_{m+2n}\| |x|^m |t|^n$$

converges for $|t| < T_0$ and $|x| < R_0$. To complete the proof, it is enough to take $\alpha_0 = c_3$, $\beta_0 = c_4$, and $\beta_1 = c_1^{1/2} c_2$. The theorem is proved. $\square$

4. Solution representation by a Poisson integral

In the classic case ($E = \mathbb{C}$ and $A > 0$), it is well known that the solution of the Cauchy problem (12) with a bounded continuous initial function $g(x)$ can be written as the Poisson integral:

$$u(t,x) = \frac{1}{2A\sqrt{\pi t}} \int_{-\infty}^{+\infty} \exp\Big\{-\frac{\xi^2}{4A^2 t}\Big\} g(x-\xi)\, d\xi.$$

In the vector case, if the operator A is noninvertible, then the expression $\frac{1}{2A\sqrt{\pi t}} \exp\{-\frac{\xi^2}{4A^2 t}\}$ has no direct sense. On the other hand, if $E = \mathbb{C}$, $A > 0$, and $g \in E[\xi]$, $g(\xi) = \sum_{m=0}^{2p} b_m \xi^m$, then it is easy to check that

$$\int_{-\infty}^{+\infty} \frac{1}{2A\sqrt{\pi t}} \exp\Big\{-\frac{\xi^2}{4A^2 t}\Big\} g(\xi)\, d\xi = \sum_{n=0}^{p} \frac{(2n)!}{n!} A^{2n} b_{2n} t^n.$$

This equality gives us a basis for the following definition of the Poisson integral in the space of formal power series.

Let E be a Banach space and let $E[[\xi]]$ be the linear space of formal power series with coefficients from E. For $r > 0$ and $g(\xi) = \sum_{k=0}^{\infty} b_k \xi^k \in E[[\xi]]$, we set

$$\|g\|_r = \sum_{k=0}^{\infty} \|b_k\| r^k, \quad E_r\langle\xi\rangle = \{g \in E[[\xi]] : \|g\|_r < +\infty\},$$

and $E\langle\xi\rangle = \bigcup_{r>0} E_r\langle\xi\rangle$. Then $(E_r\langle\xi\rangle, \|\cdot\|)$ is a Banach space, and $E\langle\xi\rangle$ is the linear space of all convergent power series with coefficients from E. We furnish $E\langle\xi\rangle$ with the topology of inductive limit of Banach spaces $E_r\langle\xi\rangle$ (see [20, Chap. 1], where the case $E = \mathbb{C}$ is considered in a similar way).

Definition 4.1. *Let $A : E \to E$ be a bounded linear operator. For $g \in E\langle\xi\rangle$ and $g(\xi) = \sum_{k=0}^{\infty} b_k \xi^k$, we define*

$$\frac{1}{2A\sqrt{\pi t}} \int_{-\infty}^{+\infty} \exp\Big\{-\frac{\xi^2}{4A^2 t}\Big\} g(\xi) d\xi = \sum_{n=0}^{\infty} \frac{(2n)!}{n!} A^{2n} b_{2n} t^n \tag{16}$$

(we consider the right-hand side of equality (16) *as an element of $E[[t]]$).*

Remark 4.2 We note that if $A = 0$, then

$$\frac{1}{2A\sqrt{\pi t}} \int_{-\infty}^{+\infty} \exp\Big\{-\frac{\xi^2}{4A^2 t}\Big\} g(\xi) d\xi = g(0).$$

Proposition 4.3. *Assume that A is quasinilpotent and that the Fredholm resolvent of A^2 is an entire function of exponential type. Then the series in the right-hand side of equality* (16) *has a positive radius of convergence. Moreover, if we define*

$$(H_A g)(t) := \frac{1}{2A\sqrt{\pi t}} \int_{-\infty}^{+\infty} \exp\Big\{-\frac{\xi^2}{4A^2 t}\Big\} g(\xi) d\xi, \tag{17}$$

then H_A is a continuous linear map from $E\langle\xi\rangle$ to $E\langle t\rangle$.

Proof. If $g(\xi) = \sum_{k=0}^{\infty} b_k \xi^k$, then

$$(H_A g)(t) = \sum_{n=0}^{\infty} \frac{(2n)!}{n!} A^{2n} b_{2n} t^n.$$

According to 2.9,

$$\frac{1}{R_{A^2}(\psi)} = \overline{\lim_{n\to\infty}} \sqrt[n]{\frac{(2n)!}{n!}\|A^{2n}\|} < +\infty.$$

Therefore,

$$\overline{\lim_{n\to\infty}} \sqrt[n]{\frac{(2n)!}{n!}\|A^{2n}\| \, \|b_{2n}\|} = \overline{\lim_{n\to\infty}} \sqrt[n]{\frac{(2n)!}{n!}\|A^{2n}\|} \sqrt[n]{\|b_{2n}\|} \le \frac{1}{R_{A^2}(\psi) R(g)^2}.$$

Thus, if $|t| < R_{A^2}(\psi)R(g)^2$, then the series in the right-hand side of equality (16) converges, i.e., $H_A g \in E\langle t\rangle$. It is obvious that H_A is linear. Let us show that H_A is continuous. To this end, we show that the all restrictions $H_A|_{E_r\langle\xi\rangle} : E_r\langle\xi\rangle \to E\langle t\rangle$, $r > 0$, are continuous. Take $r_0 > 0$ and $g \in E_{r_0}\langle\xi\rangle$. According to Proposition 2.9, $\frac{(2n)!}{n!}\|A^{2n}\| \le M^n$, $n \in \mathbb{N}$, for some $M > 0$. Let $r_1 = \frac{r_0^2}{M}$. Then

$$\|H_A g\|_{r_1} = \sum_{n=0}^{\infty} \frac{(2n)!}{n!} \|A^{2n} b_{2n,k}\| r_1^n \le \sum_{n=0}^{\infty} \|b_{2n,k}\| r_0^{2n} \le \|g\|_{r_0}.$$

Therefore, H_A is a continuous map from $E_{r_0}\langle\xi\rangle$ to $E_{r_1}\langle t\rangle$. Hence, H_A is continuous as a map from $E_{r_0}\langle\xi\rangle$ to $E\langle t\rangle$. $\square$

Theorem 4.4. *Let g be a vector-valued function that is analytic in a neighborhood of zero and $g(x) = \sum_{m=0}^{\infty} b_m x^m$, $|x| < R(g)$. Assume that A is quasinilpotent and that the Fredholm resolvent of A^2 is an entire function of exponential type. Consider T_0 and R_0 which were defined in Theorem* 3.8. *Then the solution of Cauchy problem* (12) *can be represented as*

$$u(t,x) = \frac{1}{2A\sqrt{\pi t}} \int_{-\infty}^{+\infty} \exp\Big\{-\frac{\xi^2}{4A^2 t}\Big\} g(x-\xi) d\xi, \tag{18}$$

$|t| < T_0$, $|x| < R_0$, *i.e., for each fixed* $x \in (-R_0, R_0)$, *the right-hand side of equality* (17) *is a convergent power series in* $t \in (-T_0, T_0)$ *which coincides with the series in the right-hand side of identity* (18).

Proof. Let us fix $x \in (-R_0, R_0)$ and show that $g(x-\xi)$ is a convergent power series with respect to ξ. Let $g(\xi) = \sum\limits_{m=0}^{\infty} b_m \xi^m$, $|\xi| < R(g)$. Since $R_0 < R(g)$, $R(g) - |x| > 0$. Therefore, if $|\xi| < R(g) - |x|$, then $|x| + |\xi| < R(g)$, i.e.,

$$\sum_{m=0}^{\infty} \|b_m\|(|x| + |\xi|)^m = \sum_{m=0}^{\infty} \sum_{k=0}^{m} C_m^k \|b_m\| |\xi|^k |x|^{m-k} < +\infty.$$

From here it follows that

$$g(x-\xi) = \sum_{m=0}^{\infty} b_m (x-\xi)^m = \sum_{m=0}^{\infty} \sum_{k=0}^{m} (-1)^k b_m C_m^k \xi^k x^{m-k} = \sum_{k=0}^{\infty} \Big(\sum_{m=k}^{\infty} C_m^k b_m x^{m-k} \Big) \xi^k,$$

$|\xi| < R(g) - |x|$. Thus, if $h(\xi) = g(x-\xi)$, then $h \in E\langle \xi \rangle$, i.e., the right-hand side of identity (18) is defined correctly. According to definition 4.1,

$$\frac{1}{2A\sqrt{\pi t}} \int_{-\infty}^{+\infty} \exp\Big\{ -\frac{\xi^2}{4A^2 t} \Big\} g(x-\xi) d\xi = \sum_{n=0}^{\infty} \frac{(2n)!}{n!} A^{2n} \Big(\sum_{m=2n}^{\infty} C_m^{2n} b_m x^{m-2n} \Big) t^n$$

$$= \sum_{n=0}^{\infty} \frac{(2n)!}{n!} A^{2n} \Big(\sum_{m=0}^{\infty} C_{m+2n}^{2n} b_{m+2n} x^m \Big) t^n = \sum_{n,m=0}^{\infty} \frac{(m+2n)!}{m!n!} A^{2n} b_{m+2n} t^n x^m = u(t,x),$$

and this series converges if $|t| < T_0$, $|x| < R_0$ (see Theorem 3.8). □

Remark 4.5. Equalities (17) and (18) show that a solution $u(t,x)$ of the Cauchy problem (12) can be considered as a "convolution" of the initial condition $g(x)$ with the "distribution" H_A.

5. Cauchy problem for an inhomogeneous equation

Let $f(t,x)$ be a vector-valued function that is analytic in a neighborhood of zero and

$$f(t,x) = \sum_{n,m=0}^{\infty} f_{nm} t^n x^m, \quad |t| < T_0, \quad |x| < R_0.$$

Consider the following Cauchy problem:

$$\begin{cases} \dfrac{\partial u}{\partial t} = A^2 \dfrac{\partial^2 u}{\partial^2 x} + f(t,x), \\ u(0,x) = 0. \end{cases} \tag{19}$$

By a solution of this problem we mean a vector-valued function of real variables t and x that is analytic in a neighborhood of zero, satisfies the equation in this neighborhood, and satisfies the initial condition.

Theorem 5.1. *Assume that the operator* A *is quasinilpotent and that the Fredholm resolvent* $(1 - zA^2)^{-1}$ *of* A^2 *is an entire function of exponential type. Then the Cauchy problem* (19) *has a unique analytic solution, which is defined in a rectangle* $|t| < T_1$, $|x| < R_1$, *where* $T_1 = \min\{T_0, \alpha(1-\beta)^2\gamma^2 R_{A^2}(\psi) {R_0}^2\}$, $R_1 = \beta\gamma R_0$, *and* α, β, γ *are arbitrary constants from* $(0,1)$. *(Recall that* $R_{A^2}(\psi)$ *is the radius of convergence of the series* $\psi(zA^2) = \sum\limits_{n=0}^{\infty} \frac{(2n)!}{n!} A^{2n} z^n$, *see Proposition* 2.9*).*

Proof. Let us find the solution of the Cauchy problem (19) in the following form:

$$u(t,x) = \sum_{k=0}^{\infty} A^{2k} u_k(t,x). \tag{20}$$

It is easy to check that this series formally satisfies the equation

$$\frac{\partial u}{\partial t} = A^2 \frac{\partial^2 u}{\partial x^2} + f(t,x)$$

if

$$\frac{\partial u_0}{\partial t} = f(t,x) \quad \text{and} \quad \frac{\partial u_{k+1}}{\partial t} = \frac{\partial^2 u_k}{\partial x^2}, \quad k \geq 1.$$

Taking into account the zero initial condition, we obtain the equalities

$$u_0(t,x) = \int_0^t f(\tau_0, x) d\tau_0$$

and

$$u_{k+1}(t,x) = \int_0^t \frac{\partial^2 u_k}{\partial x^2}(\tau_{k+1}, x) d\tau_{k+1}, \quad k \geq 1.$$

Hence,

$$u_k(t,x) = \int_0^t d\tau_k \int_0^{\tau_k} d\tau_{k-1} \cdots \int_0^{\tau_1} \frac{\partial^{2k} f}{\partial x^{2k}}(\tau_0, x) d\tau_0, \quad k \geq 0.$$

To prove that the formal sum (20) is an analytic solution, we consider the functions f and u_k, $k = 0, 1, \ldots,$ as holomorphic functions of two complex variables z and w in the polydisk $|z| < T_0$, $|w| < R_0$. Thus,

$$f(z,w) = \sum_{n,m=0}^{\infty} f_{nm} z^n w^m, \quad u_0(z,w) = \sum_{n,m=0}^{\infty} \frac{f_{nm}}{n+1} z^{n+1} w^m,$$

and

$$u_k(z,w) = \sum_{n,m=0}^{\infty} \frac{(m+2k)!n!}{m!(n+k+1)!} f_{nm} z^{n+k+1} w^m,$$

$k = 0, 1, ...,$ $|z| < T_0$, $|w| < R_0$. Now take α, β, γ, $\in (0,1)$, $r = \gamma R_0$, and $s < \min\{T_0, \alpha(1-\beta)^2 \gamma^2 R_{A^2}(\psi) {R_0}^2\}$. There exists a constant M_1 such that $\|f_{nm}\| \leq \frac{M_1}{s^n r^m}$, $n, m = 0, 1, ...$ Hence, if $|z| < s_1 < s$ and $|w| < r_1 = \beta r$, then

$$\begin{aligned}
\|u_k(z,w)\| &\leq M_1 \frac{|z|^{k+1}}{r^{2k}} \sum_{n,m=0}^{\infty} \frac{(m+2k)!n!}{m!(n+k+1)!} \Big(\frac{|z|}{s}\Big)^n \Big(\frac{|w|}{r}\Big)^m \\
&= M_1 (2k)! \frac{|z|^{k+1}}{r^{2k}} \sum_{m=0}^{\infty} \frac{(m+2k)!}{m!(2k)!} \Big(\frac{|w|}{r}\Big)^m \sum_{n=0}^{\infty} \frac{n!}{(n+k+1)!} \Big(\frac{|z|}{s}\Big)^n \\
&= M_1 (2k)! \frac{|z|^{k+1}}{r^{2k}} \frac{1}{(1-\frac{|w|}{r})^{2k+1}} \sum_{n=0}^{\infty} \frac{n!}{(n+k+1)!} \Big(\frac{|z|}{s}\Big)^n \\
&\leq M_1 (2k)! \frac{s^{k+1}}{r^{2k}} \frac{1}{(1-\frac{|w|}{r})^{2k+1}} \frac{1}{(1-\frac{|z|}{s})(k+1)!} = \frac{M_1 (2k)! s^{k+1}}{(1-\frac{s_1}{s})(k+1)!(1-\frac{r_1}{r})^{2k+1} r^{2k}}
\end{aligned}$$

since

$$\Big| \sum_{n=0}^{\infty} \frac{n!}{(n+k+1)!} t^n \Big| = \Big| \int_0^t d\tau_k \int_0^{\tau_k} d\tau_{k-1} \ldots \int_0^{\tau_1} \frac{d\tau_0}{1-\tau_0} \Big| \leq \frac{1}{(1-|t|)(k+1)!}.$$

Now set $l = \alpha R_{A^2}(\psi)$. Then the series $\sum_{k=0}^{\infty} \frac{(2k)!}{k!} \|A^{2k}\| l^k$ converges (see Proposition 2.9). Therefore, there exists a constant $M_2 > 0$ such that $\|A^{2k}\| \leq M_2 \frac{k!}{(2k)! l^k}$, $k = 0, 1, ...$ Hence,

$$\|A^{2k} u_k(t,x)\| \leq \|A^{2k}\| \, \|u_k(t,x)\| \leq \frac{M_1 M_2 s^{k+1}}{(1-\frac{s_1}{s})(k+1)(1-\frac{r_1}{r})^{2k+1} r^{2k} l^k}$$

for all $|z|\le s_1$, $|w|\le r_1$, and

$$\frac{s}{l(1-\frac{r_1}{r})^2}=\frac{s}{\alpha(1-\beta)^2\gamma^2 R_{A^2}(\psi){R_0}^2}<1.$$

Thus, the series $\sum_{k=0}^{\infty} A^{2k}u_k(z,w)$ converges uniformly at $|z|\le s_1$, $|w|\le r_1$ for all $s_1<s$ and $r_1<\beta R_0$, and the function $u(z,w)=\sum_{k=0}^{\infty} A^{2k}u_k(z,w)$ is holomorphic in the polydisk $|z|<T_1$, $|w|<R_1$, where

$$T_1=\min\{T_0,\alpha(1-\beta)^2\gamma^2 R_{A^2}(\psi){R_0}^2\} \quad \text{and} \quad R_1=\beta\gamma R_0.$$

Therefore, the function $u(t,x)$, which is the sum of the series (20), is analytic in the rectangle $|t|<T_1$, $|x|<R_1$, and is a solution of the Cauchy problem (19). The uniqueness of the solution follows from Lemma 3.1. The theorem is proved. $\square$

Remark 5.2. Assume that the function f from the Cauchy problem (19) does not depend on t, i.e., $f(t,x)=g(x)$, where g is a vector-valued function that is analytic in a neighborhood of zero. If $u(t,x)$ is a vector-valued function that is analytic in a neighborhood of zero and $v=\frac{\partial u}{\partial t}$, then it is easy to check that $u(t,x)$ is a solution of the Cauchy problem (19) if and only if $v(t,x)$ is a solution of the Cauchy problem (12):

$$\begin{cases} \dfrac{\partial v}{\partial t}=A^2\dfrac{\partial^2 v}{\partial x^2}, \\ v(0,x)=g(x). \end{cases}$$

Therefore, the implication (3) ⇒ (1) in Theorem 3.8 can be deduced from Theorem 5.1. Moreover, the method of solution finding for the inhomogeneous equation in the form of a series with respect to degrees of a "small parameter" can be used to solve the Cauchy problem (12):

$$\begin{cases} \dfrac{\partial u}{\partial t}=A^2\dfrac{\partial^2 u}{\partial x^2}, \\ u(0,x)=g(x). \end{cases}$$

In this case, it is natural to find a solution in the form

$$u(t,x)=g(x)+\sum_{n=1}^{\infty}\frac{t^n}{n!}A^{2n}g_n(x). \tag{21}$$

Here $g_{n+1}=g_n''(x)$, i.e., $g_n(x)=g^{(2n)}(x)$, $n\ge 1$. If the condition of Proposition 2.9 is fulfilled, then the convergence of series (21) can be proved in the same way as in Theorem 5.1.

Example 5.3. Assume that the operator A satisfies the condition of Theorem 5.1 and that $b\in E$. Consider the Cauchy problem:

$$\begin{cases} \dfrac{\partial u}{\partial t}=A^2\dfrac{\partial^2 u}{\partial x^2}+\dfrac{b}{1-x}, \\ u(0,x)=0. \end{cases}$$

If $v=\frac{\partial u}{\partial t}$, then v is a solution of the Cauchy problem (13):

$$\begin{cases} \dfrac{\partial v}{\partial t}=A^2\dfrac{\partial^2 u}{\partial x^2}, \\ v(0,x)=\dfrac{b}{1-x}. \end{cases}$$

Therefore, $v(t,x)=\psi\left(\frac{tA^2}{(1-x)^2}\right)\frac{b}{1-x}$ (see Proposition 3.2). Hence,

$$u(t,x)=\int_0^t \psi\left(\frac{\tau}{(1-x)^2}A^2\right)\frac{b}{1-x}d\tau=\sum_{k=0}^{\infty}\frac{(2k)!}{(k+1)!}A^{2k}b\frac{t^{k+1}}{(1-x)^{2k+1}},$$

$|t| < T$, $|x| < R$, where $R = \alpha$ and $T = \frac{1}{4}(1-\alpha)^2$, $\alpha \in (0,1)$ (see Proposition 3.2).

Translated by A. Vershynina and S. Gefter.

REFERENCES

1. E. Hille, *Ordinary Differential Equations in the Complex Domain*, A Wiley-Interscience publication, New York–London (1976).
2. J. Hadamar, *Le Problème de Cauchy et les Équations aux Dérivées Partielles Linéaires Hyperboliques*, Hermann, Paris (1932).
3. R. Courant, *Partial Differential Equations*, New York–London (1962).
4. I. Petrovskii, *Partial Differential Equations*, Iliffe Books Ltd., London (1967).
5. S. V. Kowalewsky, "Zur Theorie der partiellen Differentialgleichungen," *J. reine angew. Math.*, **80**, 1–32 (1875).
6. C. Riquie, *Les Systémes d'Èquations aux Dérivées Partielles*, Gauthier-Villars, Paris (1910).
7. P. Rosebloom, "The majorant method. Partial differential equations," in: *Proc. of the Fourth Symposium in Pure Mathemetics*, Amer. Math. Soc. (1961), pp. 51–72.
8. V. Palamodov, "Differential operators on the class of convergent power series and the Weierstrass auxiliary lemma," *Funkts. Analiz Prilozh.*, **2**, 235–244 (1968).
9. F. Treves, "An abstract nonlinear Cauchy–Kovalevskaya theorem," *Trans. Amer. Math. Soc.*, **150**, 77–92 (1970).
10. S. Mizohata, "On Kowalewskian systems," *Russ. Math. Surv.* **29**, 223–235 (1974).
11. L. Ovsyannikov, "Abstract form of the Cauchy–Kovalevskaya theorem and its applications," *Partial Diff. Eqs*, Novosibirsk, 88–94 (1980).
12. S.Öuchi, "Characteristic Cauchy problems and solutions of formal power series," *Ann. Inst. Fourier*, **33**, 131–176 (1983).
13. D. Lutz, M. Miyake, and R. Schäfke, "On the Borel summability of divergent solutions of the heat equation," *Nagoya Math. J.*, **154**, 1–29 (1999).
14. W. Balser, "Divergent solutions of the heat equation: on an article of Lutz, Miyake, and Schäfke," *Pacific J. Math.*, **188**, 53–63 (1999).
15. M. Gorbachuk, "On the well-posedness of the Cauchy problem for operator-differential equations in classes of analytic vector functions," *Dokl. Ros. Akad. Nauk*, **374**, 7–9 (2000).
16. M. Gorbachuk, "An operator approach to the Cauchy–Kovalevskaya theorem," *J. Math. Sci.*, **5**, 1527–1532 (2000).
17. S. Chulkov, "On the convergence of formal solutions of systems of partial differential equations," *Funkts. Analiz Prilozh.*, **39**, 215–224 (2005).
18. A. Khovanskii and S. Chulkov, "The Hilbert polynomial for systems of linear partial differential equations with analytic coefficients," *Izv. Math.*, **70**, 153–169 (2006).
19. E. Hille and R. Phillips, *Functional Analysis and Semi-Groups*, Amer. Math. Soc. Coll. Publ., **31**, Providence (1957).
20. H. Grauert and R. Remmert, *Analytische Stellenalgebren*, Springer-Verlag (1971).
21. Ju. Dalec'kii and M. Krein, *Stability of Differential Equations in a Banach Space*, Amer. Math. Soc., Providence (1974).
22. S. Gefter and V. Mokrenyuk, "The power series $\sum_{n=0}^{\infty} n!z^n$ and holomorphic solutions of some differential equations in a Banach space," *J. Math. Phys., Anal., Geom.*, **1**, 53–70 (2005).