Methods of Functional Analysis and Topology
Vol. 13 (2007), no. 4, pp. 329–332

ON HOLOMORPHIC SOLUTIONS OF THE HEAT EQUATION WITH A VOLTERRA OPERATOR COEFFICIENT

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This paper is dedicated to 100 anniversary of Mark Krein.

Abstract. Let A be a bounded operator on a Hilbert space and g a vector-valued function, which is holomorphic in a neighborhood of zero. The question about existence of holomorphic solutions of the Cauchy problem $\begin{cases} \dfrac{\partial u}{\partial t} = A\dfrac{\partial^2 u}{\partial x^2} \\ u(0,x) = g(x) \end{cases}$ is considered in the paper.

In 1875 Sofya Kovalevskaya showed that the heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

with a holomorphic initial condition can have no holomorphic solution (see [9], [11]). The researches, started by Kovalevskaya, further have been continued in numerous works (see, for example, [6], [7], [10], [12]–[14]). In the present paper we shall consider the following operator analog of the Cauchy problem for the heat equation:

$$\begin{cases} \dfrac{\partial u}{\partial t} = A\dfrac{\partial^2 u}{\partial x^2}, \\ u(0,x) = g(x), \end{cases} \tag{1}$$

where A is a bounded linear operator in a complex Hilbert space and $g(x)$ a vector-valued function, which is holomorphic in a neighborhood of zero. As a solution of the problem (1) we understand a holomorphic solution, i.e., a vector-valued function of the two complex variables t and x, which is holomorphic in a neighborhood of zero, satisfies the equation in this neighborhood and the initial condition holds in some neighborhood of the point $x_0 = 0$. The main result of the paper is a proof of local existence and uniqueness theorem on holomorphic solution of the Cauchy problem (1) in the assumption that A is a Volterra operator and the imaginary part of A is of trace class (see Theorem 2). Let us note that operators of this class were been studied in detail in the Odessa school of the operator theory (see [2], [4], [5]). The proof of Theorem 2 is based on the consideration of the formal solution of the Cauchy problem (1)

$$u(t,x) = g(x) + \sum_{n=1}^{\infty} \frac{t^n}{n!} A^n g^{(2n)}(x)$$

as a series on degrees of the “small parameter” A.

Let H be a complex Hilbert space and $A : H \to H$ a bounded linear operator.

2000 *Mathematics Subject Classification.* 35R20.
Key words and phrases. Heat equation, Volterra operator, trace class operator, holomorphic solution.

Lemma 1. *Let* $g(x) = \sum_{m=0}^{\infty} b_m x^m$ *be a formal power series with coefficients in* E. *Then the formal power series*

$$u(t,x) = \sum_{n,m=0}^{\infty} \frac{(m+2n)!}{m!n!} A^{2n} b_{m+2n} t^n x^m$$

is a unique formal solution of the problem (1). (*We set* $(Av)(t,x) = \sum_{n,m=0}^{\infty} Av_{n,m} t^n x^m$, *if* $v(t,x) = \sum_{n,m=0}^{\infty} v_{n,m} t^n x^m$ *is a formal power series with coefficients in* H.)

Proof. Assume that $u(t,x) = \sum_{n,m=0}^{\infty} c_{nm} t^n x^m$, $c_{nm} \in H$, is a formal solution of the Cauchy problem (1). Then after substitution into the equation we obtain

$$\sum_{n,m=0}^{\infty} (n+1) c_{n+1m} t^n x^m = \sum_{n,m=0}^{\infty} (m+2)(m+1) A c_{nm+2} t^n x^m,$$

$$\sum_{m=0}^{\infty} c_{0m} x^m = \sum_{m=0}^{\infty} b_m x^m.$$

Hence,

$$(n+1) c_{n+1m} = (m+2)(m+1) A c_{nm+2}$$

and

$$c_{0m} = b_m, \quad n, m \geq 0,$$

i.e.,

$$c_{n,m} = \frac{(m+2n)!}{m!n!} A^n b_{m+2n}, \quad n, m \geq 0.$$

Thus the unique formal solution is $u(t,x) = \sum_{n,m=0}^{\infty} \frac{(m+2n)!}{m!n!} A^n b_{m+2n} t^n x^m$. □

Theorem 1. *Let the operator* A *be normal (in particular, self-adjoint). If the Cauchy problem (1) has a holomorphic solution for any vector-valued function* $g(x)$, *which is holomorphic in a neighborhood of zero, then* $A = 0$.

Proof. Since the problem (1) has a holomorphic solution for any vector-valued function $g(x)$, the Cauchy problem

$$\begin{cases} \dfrac{\partial u}{\partial t} = A \dfrac{\partial^2 u}{\partial x^2}, \\ u(0,x) = \dfrac{b}{1-x}, \end{cases}$$

has a holomorphic solution for all vector $b \in H$. Let $u(t,x)$ be a solution of this problem. It follows from Lemma 1 that $u(t,x) = \sum_{n,m=0}^{\infty} \frac{(m+2n)!}{m!n!} A^n b t^n x^m$. Since $\sum_{m=0}^{\infty} \frac{(m+2n)!}{m!(2n)!} x^m = \frac{1}{(1-x)^{2n+1}}$, we obtain that $u(t,x) = \sum_{n=0}^{\infty} \frac{(2n)!}{n!} \frac{A^n b t^n}{(1-x)^{2n+1}}$. Therefore $\overline{\lim_{n\to\infty}} \sqrt[n]{\frac{(2n)!}{n!} \|A^n b\|} < +\infty$. Since $\sqrt[n]{\frac{(2n)!}{n!}} \to \infty$, we obtain that $\lim_{n\to\infty} \sqrt[n]{\|A^n b\|} = 0$. Hence, $\lim_{n\to\infty} \sqrt[n]{\|A^n\|} = 0$ (see, for example, [1], Problem 6.1.10). Since A is normal, $A = 0$. □

Theorem 2. *Let the operator* A *be quasi-nilpotent and its imaginary part* $A_I := \frac{1}{2i}(A - A^*)$ *be of trace class. Then the Cauchy problem (1) has a unique holomorphic solution in a neighborhood of zero.*

Proof. We shall seek a solution of the Cauchy problem (1) in the form

$$u(t,x)=\sum_{n=0}^{\infty}A^n u_n(t,x). \tag{2}$$

It is easy to check that this series formally satisfies the equation $\frac{\partial u}{\partial t}=A\frac{\partial^2 u}{\partial x^2}$ if

$$\frac{\partial u_0}{\partial t}=0 \quad \text{and} \quad \frac{\partial u_{n+1}}{\partial t}=\frac{\partial^2 u_n}{\partial x^2}, \quad n\geq 0.$$

Taking into account the initial condition, we obtain

$$u_0(t,x)=g(x) \quad \text{and} \quad u_{n+1}(t,x)=\int_0^t \frac{\partial^2 u_n}{\partial x^2}(\tau,x)\,d\tau, \quad n\geq 0.$$

Hence,

$$u_n(t,x)=g^{(2n)}(x)\frac{t^n}{n!}, \quad n\geq 0.$$

Therefore, $u(t,x)=\sum_{n=0}^{\infty}\frac{1}{n!}A^n g^{(2n)}(x)t^n$. Now we show that there exist T_0, $R_0>0$ such that this series converges uniformly in $|t|<T_0, |x|<R_0$. Let $g(x)=\sum\limits_{m=0}^{\infty}b_m x^m, |x|<R(g)$. Let $c_1,c_2,c_3,c_4\in(0,1)$ and $c_3<c_1^2c_2(1-c_4)^2$ (for example $c_1=3/4, c_2=9/10, c_3=1/8, c_4=1/2$). If $0<r_1=c_1R(g)$, then there exists a constant $M_1>0$ such that $\|b_m\|\leq\frac{M_1}{r_1^m}$, $m=0,1,\dots$. Hence, for $|x|<r_1$ and $n=0,1,\dots$, we obtain

$$\begin{aligned}\|g^{(2n)}(x)\|=\|\sum_{m=0}^{\infty}\frac{(m+2n)!}{m!}b_{m+2n}x^m\|\leq\sum_{m=0}^{\infty}\frac{(m+2n)!}{m!}\|b_{m+2n}\||x|^m\\ \leq\frac{M_1}{r_1^{2n}}\sum_{m=0}^{\infty}\frac{(m+2n)!}{m!}(\frac{|x|}{r_1})^m=\frac{M_1(2n)!}{r_1^{2n}(1-\frac{|x|}{r_1})^{2n+1}}.\end{aligned}$$

Therefore,

$$\sum_{n=0}^{\infty}\|\frac{1}{n!}A^n g^{(2n)}(x)t^n\|\leq M_1\sum_{n=0}^{\infty}\frac{(2n)!}{n!}\|A^n\|\frac{(\frac{|t|}{r_1^2})^n}{(1-\frac{|x|}{r_1})^{2n+1}}.$$

Let us consider the Fredholm resolvent $F_A(z):=(1-zA)^{-1}$ of A. Since A is quasi-nilpotent, $F_A(z)$ is an entire function. Moreover, A is quasi-nilpotent and A_I is compact. Hence A is compact (see [5], Ch. 1, Th. 5.4). So, A is a Volterra operator. Now from ([5], Ch. 4, Rem. 8.3 and Ch. 5, Th. 5.2), we obtain that $F_A(z)$ is of exponential type, i.e., $\overline{\lim\limits_{z\to\infty}}\frac{\ln\|F_A(z)\|}{|z|}<+\infty$. Hence, $\overline{\lim\limits_{n\to\infty}}\sqrt[n]{\|A^n\|}<+\infty$ (see [3], Ch. 1, Problem 22). It follows from the Stirling formula that $\frac{1}{R_A}:=\overline{\lim\limits_{n\to\infty}}\sqrt[n]{\frac{(2n)!}{n!}\|A^n\|}<+\infty$. Hence, the series $\sum\limits_{n=0}^{\infty}\frac{(2n)!}{n!}\|A^n\|r_2^n$ converges, where $r_2=c_2R_A$. Therefore, there exists a constant $M_2>0$ such that $\frac{(2n)!}{n!}\|A^n\|\leq\frac{M_2}{r_2^n}$, $n=0,1,\dots$ Hence, for $|x|<r_1$ and $n=0,1,\dots$, we obtain

$$M_1\frac{(2n)!}{n!}\|A^n\|\frac{(\frac{|t|}{r_1^2})^n}{(1-\frac{|x|}{r_1})^{2n+1}}\leq\frac{M_1M_2}{r_1^{2n}r_2^n}\frac{|t|^n}{(1-\frac{|x|}{r_1})^{2n+1}}.$$

Let now $T_0=c_3R_AR(g)^2,\ R_0=c_1c_4R(g)$. If $|t|<T_0$ and $|x|<R_0$, then

$$\frac{|t|}{r_1^2r_2(1-\frac{|x|}{r_1})^2}<\frac{c_3R_AR(g)^2}{c_2R_Ac_1^2R(g)^2(1-\frac{R_0}{r_1})^2}=\frac{c_3}{c_1^2c_2(1-c_4)^2}<1,$$

i.e., the series $\sum_{n=0}^{\infty} \frac{1}{n!} A^n g^{(2n)}(x) t^n$ converges uniformly in the dicylinder $|t| < T_0$, $|x| < R_0$. Hence, $u(t,x) = \sum_{n=0}^{\infty} \frac{1}{n!} A^n g^{(2n)}(x) t^n$ is holomorphic in this dicylinder and $u(t,x)$ is a solution of the problem (1). The uniqueness follows from Lemma 1. □

Acknowledgments. The authors are grateful to Vladimir Dubovoj, Anatolij Rutkas, Leonid Vaksman, Larisa Vlasenko and Vladimir Zolotarev for useful discussions.

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Received 03/01/2007