From Fourier to Wavelets in 60 Slides

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Outline

1. From Fourier to Filters
   - Inner product spaces
   - Fourier series
   - Fourier transform
   - Sampling and reconstruction
   - Convolution and filters
   - From analog to digital filters
2. Wavelets
   - Haar wavelets
   - Multiresolution Analysis
   - The scaling relation
   - Properties of the scaling function and the wavelet
   - Decomposition and reconstruction
   - Wavelet design in the frequency domain
   - The Daubechies wavelet
   - Construction of the Daubechies wavelet

Why wavelets?

The scoop about wavelets:

- Similar filtering capabilities as Fourier series/transform.
- Adaptability to typical signals.
- Good for compression and denoising.
- Numerical implementation as fast as FFT.
- Wavelet design based on digital filters.

More information about analog-digital conversion, filtering, wavelet design and applications in MATH 4355, “Mathematics of Signal Representations” (From Fourier to Wavelets in 1 Semester) in Spring 2009!

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Inner Product Spaces

The typical examples of vector spaces with an inner product are given by *sequences* or by *functions*. A fundamental relationship between vectors in inner product spaces is orthogonality.

Definition

Let $\ell^2(\mathbb{Z})$ be the vector space of all (bi-infinite) sequences $(x_n)_{n\in\mathbb{Z}}$ with $\sum_{k=-\infty}^{\infty} |x_n|^2 < \infty$. For $x, y \in \ell^2(\mathbb{Z})$, we define an **inner product**

$$\langle x, y\rangle = \sum_{n=-\infty}^{\infty} x_n \overline{y_n}\,.$$

This means, we can measure the "length" of a square-summable sequence x, the **norm** $\|x\| = \sqrt{\langle x, x\rangle}$ and an "angle" θ between two non-zero sequences x and y by

$$\cos\theta = |\langle x, y\rangle| / \|x\|\|y\|\,.$$

We call them **orthogonal** if $\langle x, y\rangle = 0$.

Example

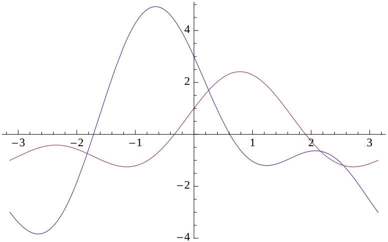
An example for an inner product space of functions is given by all trigonometric polynomials, convention: $i = \sqrt{-1}$,

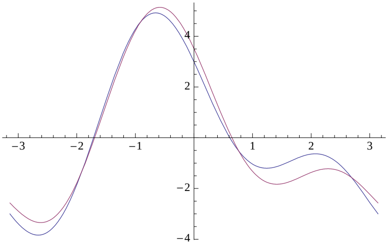
$$V = \left\{ p : [0,1] \to \mathbb{C}, p(t) = \sum_{k=-N}^{N} c_k e^{2\pi i k t}, N \in \mathbb{N},\ \text{all } c_k \in \mathbb{C} \right\} ,$$

equipped with the inner product

$$\langle v, w \rangle = \int_0^1 v(t)\overline{w(t)}\, dt \, .$$

An orthogonal pair of (real-valued) polynomials





A nearly collinear pair of polynomials

Because of the orthogonality of complex exponentials, the inner product of two trigonometric polynomials v and w is expressed in terms of their coefficients $(c_k)_{k\in\mathbb{Z}}$ and $(d_k)_{k\in\mathbb{Z}}$ as

$$\langle v, w \rangle = \sum_{k\in\mathbb{Z}} c_k \overline{d_k} \, .$$

The space of sequences can be thought of as the space of digitized signals, given by coefficients stored in a computer. The function space of trig polynomials, on the other hand, can be thought of as a space of analog signals. We have just converted the inner product from an integral to a series, *from analog to digital*, without changing it!

$L^2([a, b])$

We can make a more general type of function space by linear combinations of complex exponentials of the form $e^{2\pi int/(b-a)}$.

Definition

Let $a, b \in \mathbb{R}$, $a < b$, then we define

$$L^2([a, b]) = \left\{ f : [a, b] \to \mathbb{C}, f(t) = \sum_{k=-\infty}^{\infty} c_k e^{2\pi ikt/(b-a)}, c \in \ell^2(\mathbb{Z}) \right\}$$

and for two such square-integrable functions f and g, we write

$$\langle f, g \rangle = \int_a^b f(t)\overline{g(t)} dt \,.$$

Again, the inner product of f and g can be rewritten as inner product of their coefficients in $\ell^2(\mathbb{Z})$.

Orthogonality and basis expansions

Definition

Let V be a vector space with an inner product. A set $\{e_1, e_2, \ldots e_N\}$ is called **orthonormal** if $\|e_i\| = 1$ and $\langle e_i, e_j \rangle = 0$ for all $i \neq j$.
We call $\{e_1, e_2, \ldots e_N\}$ an **orthonormal basis** for its linear span.
Given an infinite orthonormal set $\{e_n\}_{n \in \mathbb{Z}}$, we say that it is an orthonormal basis for all vectors obtained from summing the basis vectors with square-summable coefficients.

Definition

Two subspaces V_1, V_2 are called **orthogonal**, abbreviated $V_1 \perp V_2$, if all pairs (x, y) with $x \in V_1$ and $y \in V_2$ are orthogonal.

We consider two examples of subspaces of $L^2([a,b])$:

Example

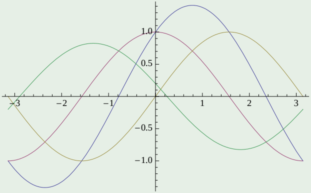
Let V_0 be the complex subspace of $L^2([-\pi,\pi])$ given by

$$V_0 = \{f(x) = c_1 \cos x + c_2 \sin x \text{ for } c_1, c_2 \in \mathbb{C}\}.$$

Then the set $\{e_1, e_2\}$,

$$e_1(x) = \frac{1}{\sqrt{\pi}} \cos x \text{ and } e_2(x) = \frac{1}{\sqrt{\pi}} \sin x,$$

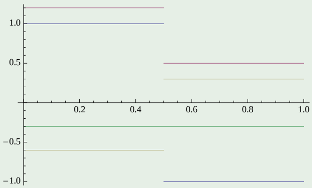
is an orthonormal basis for V_0.



Example

Another subspace of of $L^2([0,1])$ is the space of functions which are (almost everywhere) constant on $[0,1/2)$ and $[1/2,1]$. It has the orthonormal basis $\{\phi,\psi\}$ with

$$\phi(x) = 1 \text{ and } \psi(x) = \begin{cases} 1, & 0 \le x < 1/2 \\ -1, & 1/2 \le x \le 1 \end{cases}$$



Such finite-dimensional subspaces of $L^2([a,b])$ are often chosen to specify approximations of signals.

Once we have orthonormal bases, we can use them to expand vectors.

Theorem

Let V_0 be a subspace of an inner product space V, and $\{e_1, e_2, \dots e_N\}$ an orthonormal basis for V_0. Then for all $v \in V_0$,

$$v = \sum_{k=1}^{N} \langle v, e_k \rangle e_k \, .$$

Question

What is the result

$$\hat{v} = \sum_{k=1}^{N} \langle v, e_k \rangle e_k$$

if $v \notin V_0$?

It turns out that $\hat{v}$ is the best you can get with a linear combination from $\{e_1, e_2, \dots e_N\}$.

Orthogonal projections

Theorem

Let V_0 *be an inner product space,* V_0 *an* N*-dimensional subspace with an orthonormal basis* $\{e_1, e_2, \dots e_N\}$*. Then for* $v \in V$*,*

$$\hat{v} = \sum_{j=1}^{N} \langle v, e_k \rangle e_k$$

satisfies

$$\langle v - \hat{v}, w_0 \rangle = 0$$

for all $w_0 \in V_0$*.*

Since the difference vector $v - \hat{v}$ is orthogonal to V_0, we call $\hat{v}$ the **orthogonal projection** of v onto V_0.

A least squares property

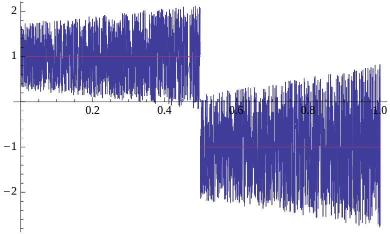
Here is why the vector $\hat{v}$ is the "best possible" choice in the subspace V_0:

Theorem

Let V_0 be a finite-dimensional subspace of an inner product space V. Then for any $v \in V$, its orthogonal projection $\hat{v}$ onto V_0 has the least-squares property

$$\|v - \hat{v}\|^2 = \min_{w \in V_0} \|v - w\|^2 .$$

Projection onto subspace of piecewise constant functions in $L^2([0,1])$



Fourier series as expansion in an orthonormal basis

Exercise

Given $V_0 \subset L^2([0,\pi])$ which has the orthonormal basis $\{e_k\}_{k=1}^N$ of functions $e_k(x) = \sqrt{\frac{2}{\pi}}\sin(kx)$. Compute the projection of the constant function $f(x) = C$, $C \in \mathbb{R}$, onto V_0.

This exercise amounts to computing the first N terms in the Fourier sine expansion of f! We can include cosines, choose the interval $[-\pi, \pi]$ and take $N \to \infty$ in order to get the Fourier series.

Theorem

The set $\{\dots, \frac{\cos(2x)}{\sqrt{\pi}}, \frac{\cos(x)}{\sqrt{\pi}}, \frac{1}{\sqrt{2\pi}}, \frac{\sin(x)}{\sqrt{\pi}}, \frac{\sin(2x)}{\sqrt{\pi}}, \dots\}$ *is an orthonormal set in* $L^2([-\pi,\pi])$.

Theorem

If a function is given as a series,

$$f(x) = a_0 + \sum_{k=1}^{\infty} (a_k \cos(kx) + b_k \sin(kx))$$

which converges with respect to the norm in $L^2([-\pi, \pi])$*, then*

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) dx \,,$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx) dx \,,$$

and

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx) dx \,.$$

Convergence of Fourier series

Theorem

Let f be square integrable on $[-\pi, \pi]$, then the partial sums of the Fourier series

$$S_N(x) = a_0 + \sum_{k=1}^{N} (a_k \cos(kx) + b_k \sin(kx))$$

converge in square mean to f,

$$\lim_{N \to \infty} \int_{-\pi}^{\pi} |(f - S_N)(x)|^2 dx = 0 \, .$$

Remark

This is just convergence in the norm of $L^2([-\pi, \pi])$. Other types of convergence can also be proved, but they require more assumptions.

Fact

If $f \in L^2(\mathbb{R})$, then

$$\hat{f}(\omega) = \lim_{L \to \infty} \frac{1}{\sqrt{2\pi}} \int_{-L}^{L} f(t) e^{-i\omega t} dt$$

exists for almost all $\omega \in \mathbb{R}$, that is, up to a set which does not count under the integral. Moreover, $\hat{f} \in L^2(\mathbb{R})$ and

$$f(t) = \lim_{\Omega \to \infty} \frac{1}{\sqrt{2\pi}} \int_{-\Omega}^{\Omega} \hat{f}(\omega) e^{i\omega t} d\omega \, ,$$

again, up to a set of $t \in \mathbb{R}$ which does not count in integrals.

When a function $f \in L^2([-\pi, \pi])$ is expanded in an orthonormal basis $\{e_j\}_{j\in\mathbb{Z}}$, $f = \sum_{j\in\mathbb{Z}} \langle f, e_j \rangle e_j$, Pythagoras gives

$$\|f\|^2 = \sum_{j\in\mathbb{Z}} |\langle f, e_j \rangle|^2.$$

A similar statement is true for the Fourier transform.

Theorem (Plancherel)

Let $f \in L^2(\mathbb{R})$, then denoting $F[f] = \hat{f}$, we have

$$\|F[f]\|^2 = \|f\|^2 .$$

The same is true for inner products of $f, g \in L^2(\mathbb{R})$,

$$\langle f, g \rangle = \langle F[f], F[g] \rangle .$$

Transforming a signal from time to frequency domain preserves geometry (inner products)!

Proposition

Let $f, h \in L^2(\mathbb{R})$, $h(t) = f(bt)$ *for* $b > 0$. *Then* $\hat{h}(\omega) = \frac{1}{b}\hat{f}(\frac{\omega}{b})$.

Example

If

$$f(t) = \begin{cases} 1, & -\pi \le t \le \pi \\ 0, & \text{else} \end{cases}$$

then $h(t) = f(bt)$ has the Fourier transform

$$\hat{h}(\omega) = \sqrt{\frac{2}{\pi}} \frac{\sin(\pi\omega/b)}{\omega} .$$

Proposition

Let $f, h \in L^2(\mathbb{R})$, $h(t) = f(t-a)$ *for some* $a \in \mathbb{R}$. *Then* $\hat{h}(\omega) = e^{-i\omega a}\hat{f}(\omega)$.

Sampling and reconstruction

Definition

A function $f \in L^2(\mathbb{R})$ is called **Ω-bandlimited** if $\hat{f}(\omega) = 0$ for almost all ω with $|\omega| > \Omega$.

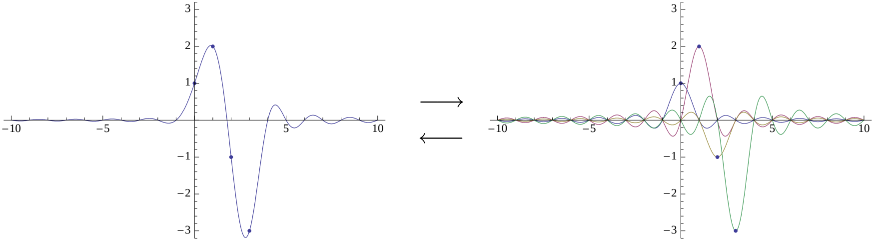
Theorem

Let $f \in L^2(\mathbb{R})$ be Ω-bandlimited, then it is continuous and

$$f(t) = \sum_{k=-\infty}^{\infty} f\left(\frac{k\pi}{\Omega}\right) \frac{\sin(\Omega t - k\pi)}{\Omega t - k\pi}$$

and the series on the right-hand side converges in the norm of $L^2(\mathbb{R})$ and uniformly on $\mathbb{R}$.

Sampling and reconstruction in action



Convolutions and filters

Definition

Let $f, g \in L^2(\mathbb{R})$. Then we denote the **convolution** of f and g by

$$(f * g)(t) = \int_{-\infty}^{\infty} f(t-x)g(x)dx\,.$$

Example

Take

$$g(x) = \begin{cases} 1/a, & 0 \le x \le a \\ 0, & \text{else} \end{cases}$$

then for any integrable (or square-integrable) f,

$$(f * g)(t) = \int_0^a f(t-x)dx = \int_{t-a}^{t} f(x)dx\,.$$

Theorem

Let f, g be integrable functions on $\mathbb{R}$. Then $f * g$ is again integrable and $F[f * g] = \sqrt{2\pi}\hat{f}\hat{g}$.
If, in addition $f, g \in L^2(\mathbb{R})$, then $F^{-1}[\hat{f}\hat{g}] = \frac{1}{\sqrt{2\pi}} f * g$.

Remark

Convolving f with an integrable function g on $\mathbb{R}$ amounts to multiplying the Fourier transform $\hat{f}$ with $\sqrt{2\pi}\hat{g}$.

Definition

A **filter** on $L^2(\mathbb{R})$ is a linear map $L : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ for which there is a bounded function m on $\mathbb{R}$ such that for all $f \in L^2(\mathbb{R})$,

$$F[Lf] = m\hat{f}\,,$$

or equivalently,

$$Lf = F^{-1}[m\hat{f}]\,.$$

The function m is called the **system function** of the filter.

Exercise

Find the system function m for the filter

$$Lf(t) = \frac{1}{2}(f(t) + f(t-a))\,, \quad a \in \mathbb{R}\,.$$

From analog to digital filters

We now examine filtering for bandlimited signals.

Question

Given a filter with a system function m and a bandlimited function f, can we express the sampled values of Lf in terms of those of f?

Remark

For filtering an Ω-bandlimited function, only the restriction of the system function m to $[-\Omega, \Omega]$ matters, because $\hat{f}$ vanishes outside of this interval. We can thus expand m in a Fourier series,

$$m(\omega) = \sum_{k \in \mathbb{Z}} \alpha_k e^{-i\pi k\omega/\Omega},$$

where we have changed the sign in the exponent because it is a series for a function on the frequency domain.

Theorem

Given an Ω-bandlimited function f and a filter L with system function m whose restriction to $[-\Omega, \Omega]$ has Fourier coefficients $\{\alpha_k\}_{k\in\mathbb{Z}}$. Then

$$Lf(\frac{k\pi}{\Omega}) = \sum_{l=-\infty}^{\infty} f(\frac{(k-l)\pi}{\Omega})\alpha_l \,.$$

The upshot is that the convolution is replaced by a series formula for the sampled values of f.

Definition

For two sequences $x, y \in l^2(\mathbb{Z})$, we define the **discrete convolution** as

$$(x * y)_k = \sum_{l=-\infty}^{\infty} x_l y_{k-l} \,.$$

Oversampling

Since any function with band limit Ω is also $a\Omega$-bandlimited for any $a \geq 1$, we have a generalization of the sampling theorem

Proposition

Let f be square-integrable and Ω-bandimited. Then

$$f(t) = \sum_{k=-\infty}^{\infty} f\left(\frac{k\pi}{a\Omega}\right) \frac{\sin(a\Omega t - k\pi)}{a\Omega t - k\pi} .$$

If we now apply a filter which has a system function such that

$$m(\omega) = 1 \text{ if } |\omega| \leq \Omega$$

then

$$f(t) = Lf(t) = \sum_{k=-\infty}^{\infty} f\left(\frac{k\pi}{a\Omega}\right)(L\,\text{sinc})(a\Omega t - k\pi)$$

where $\text{sinc}(t) = \frac{\sin(t)}{t}$.

Since the requirement on the system function of L only concerns the interval $[-\Omega, \Omega]$ there is freedom in choosing L and thus the resulting function for reconstructing the signal! This can be used to improve the convergence of the series used for reconstruction.

Example

Let f be square-integrable and Ω-bandimited. Then

$$f(t) = \sum_{k=-\infty}^{\infty} f(\frac{k\pi}{a\Omega}) \frac{\cos(\Omega(t - k\pi/a\Omega)) - \cos(a\Omega(t - k\pi/a\Omega))}{a(a-1)\Omega^2(t - k\pi/a\Omega)^2} .$$

Note that for fixed t, the series decays as k^{-2}!

Outline

Functions that are constant on all intervals $[n, n+1)$, $n \in \mathbb{Z}$, can be written as

$$f(x) = \sum_{k=-\infty}^{\infty} a_k \phi(x-k)$$

where

$$\phi(x) = \begin{cases} 1, & 0 \leq x < 1 \\ 0, & \text{else} \end{cases} .$$

Definition

We define the space of square-integrable integer-wide step functions as

$$V_0 = \{f(x) = \sum_{k=-\infty}^{\infty} a_k \phi(x-k), a \in \ell^2(\mathbb{Z})\} .$$

Adding details

Exercise

Show that the translates $\{\phi(\cdot - k)\}_{k\in\mathbb{Z}}$ form an orthonormal basis for V_0.

Question

Knowing the values of a function f at one point in each interval $[k, k+1)$ determines the function completely. How can we have a function space with more details?

Answer

Take $\{2^{j/2}\phi(2^j x - k)\}_{k\in\mathbb{Z}}$ as an orthonormal basis instead of $\{\phi(x-k)\}_{k\in\mathbb{Z}}$.

Levels of resolution

Definition

The space of square-integrable step functions of width 2^{-j}, denoted by V_j, is the subspace of $L^2(\mathbb{R})$ with the orthonormal basis

$$\{2^{j/2}\phi(2^j x - k)\}_{k\in\mathbb{Z}} \,.$$

Remark

Functions in this space have possible discontinuities at $x = 2^{-j}k$, $k \in \mathbb{Z}$. This implies that sampling the function values at 2^j evenly-spaced points in the interval $[k, k+1)$ determines the function on this interval.
We also note that for $j > 0$, we have the inclusions $V_{-j} \subset V_{-j+1} \subset \cdots \subset V_0 \subset V_1 \subset \cdots \subset V_{j-1} \subset V_j \subset V_{j+1}$.

Proposition

For any square integrable function f, $f \in V_0$ if and only if $f(2^j x) \in V_j$, or equivalently $f \in V_j$ if and only if $f(2^{-j} x) \in V_0$.

Question

Is there an orthonormal basis for layers of detail? We would like to have a basis of translates for $V_1 \cap V_0^\perp$.

Try

$$\psi(x) = \phi(2x) - \phi(2x - 1)$$

then

$$\int_{-\infty}^{\infty} \phi(x)\psi(x)dx = \int_0^{1/2} 1dx - \int_{1/2}^{1} 1dx = 0$$

and because ψ is supported in $[0, 1]$, it is orthogonal to all $\phi(x - k)$!

Detail spaces

Indeed, the translates of ψ form a basis for the detail spaces that bridge between V_0 and V_1. More generally, we can define a subspace of V_{j+1} which is orthogonal to V_j.

Theorem

Let W_j be the span of all functions in $L^2(\mathbb{R})$ such that

$$f(x) = \sum_{k \in \mathbb{Z}} a_k \psi(2^j x - k)\,.$$

Then $W_j = V_j^\perp \cap V_{j+1}$.

Haar decomposition

Stripping layers of detail

Now we can perform a recursive splitting. Each $f_j \in V_j$ is expressed uniquely as the sum

$$f_j = w_{j-1} + f_{j-1}$$

where $w_{j-1} \in W_{j-1}$ and $f_{j-1} \in V_{j-1}$. This orthogonal splitting is abbreviated by

$$V_j = W_{j-1} \oplus V_{j-1} \, .$$

Iterating the splitting gives

$$V_j = W_{j-1} \oplus W_{j-2} \oplus V_{j-2}$$

and so on.

If we let $j \to \infty$ and keep the last term in this direct sum decomposition fixed, say V_0, then we obtain a unique representation of each vector as a series of vectors from W_j, $j \geq 0$, and V_0.

Theorem

For each $f \in L^2(\mathbb{R})$, denote by w_j the orthogonal projection of f onto W_j. Then

$$f = f_0 + \sum_{j=0}^{\infty} w_j$$

with vectors that are orthogonal and a series that converges in norm. In short,

$$L^2(\mathbb{R}) = V_0 \oplus W_0 \oplus W_1 \oplus W_2 \oplus \cdots$$

Question

Suppose we have $f_j(x) = \sum_{k\in\mathbb{Z}} a_k \phi(2^j x - k)$, given by the values $\{a_k\}$. How do we compute the coefficients with respect to the orthonormal basis of V_j given by

$$\{\phi(x-k)\}_{k\in\mathbb{Z}} \text{ and } \{2^{l/2}\psi(2^l x - k)\}_{k\in\mathbb{Z}, 0\le l\le j-1}\,?$$

Lemma

For the Haar scaling function ϕ and the wavelet ψ,

$$\phi(2^j x) = \frac{1}{2}(\psi(2^{j-1}x) + \phi(2^{j-1}x))$$

and

$$\phi(2^j x - 1) = \frac{1}{2}(\phi(2^{j-1}x) - \psi(2^{j-1}x))\,.$$

So we can use this to convert $\sum_k a_k \phi(2^j x - k) \in V_j$ into $\sum_k (c_k \phi(2^{j-1}x - k) + d_k \psi(2^{j-1}x - k))$.

Theorem

Given a square integrable function $f_j(x) = \sum_k a_k^{(j)} \phi(2^j x - k)$ *then*

$$f_j(x) = \sum_k b_k^{(j-1)} \psi(2^{j-1}x - k) + \sum_k a_k^{(j-1)} \phi(2^{j-1}x - k)$$

with

$$b_k^{(j-1)} = \frac{a_{2k}^{(j)} - a_{2k+1}^{(j)}}{2}$$

and

$$a_k^{(j-1)} = \frac{a_{2k}^{(j)} + a_{2k+1}^{(j)}}{2} .$$

We note that both of these expressions are obtained from a digital filter applied to $\{a_l^{(j)}\}_{l\in\mathbb{Z}}$. We can repeat this procedure iteratively to obtain a coefficient tree containing $\{b_k^{(j)}\}_{k\in\mathbb{Z}}$, $\{b_k^{(j-1)}\}_{k\in\mathbb{Z}}$, $\{b_k^{(j-2)}\}_{k\in\mathbb{Z}}$, . . . and finally $\{a_k^{(0)}\}_{k\in\mathbb{Z}}$.

Reconstruction

Question

Can we reverse this procedure, that is, reconstruct the coefficients $\{a_k^{(j)}\}$ from $\{b_k^{(j-1)}\}_{k\in\mathbb{Z}}$ $\{b_k^{(j-2)}\}_{k\in\mathbb{Z}}$, ..., $\{b_k^{(0)}\}_{k\in\mathbb{Z}}$ and $\{a_k^{(0)}\}_{k\in\mathbb{Z}}$ in this coefficient tree?

Answer

We can reverse the decomposition by

$$a_{2k}^{(j+1)} = a_k^{(j)} + b_k^{(j)}$$

and

$$a_{2k+1}^{(j+1)} = a_k^{(j)} - b_k^{(j)}$$

and iterate this procedure.

Filters and up/downsampling

Definition

For any sequence $\{x_k\}_{k\in\mathbb{Z}}$ and $\{h_k\}_{k\in\mathbb{Z}}$, both in $\ell^2(\mathbb{Z})$, we define the **digital/discrete filter** of x by

$$(Hx)_k = (h * x)_k = \sum_{n\in\mathbb{Z}} x_{k-n} h_n \,.$$

Definition

The **downsampling operator** D acts on a square-summable sequence $\{x_k\}_{k\in\mathbb{Z}}$ by

$$(Dx)_k = x_{2k} \,.$$

With these two operations, we can express the analysis and reconstruction algorithm.

Remark

Let

$$h = (\ldots, 0, 0, \ldots, 0, -\frac{1}{2}, \underbrace{\frac{1}{2}}_{k=0}, 0, 0, \ldots)$$

and let

$$l = (\ldots, 0, 0, \ldots, 0, \frac{1}{2}, \underbrace{\frac{1}{2}}_{k=0}, 0, 0, \ldots)\,.$$

Then

$$(Hx)_k = (h * x)_k = \frac{1}{2}x_k - \frac{1}{2}x_{k+1}$$

and

$$(Lx)_k = (l * x)_k = \frac{1}{2}x_k + \frac{1}{2}x_{k+1}\,.$$

Remark

Therefore,

$$b_k^{(j-1)} = \frac{1}{2}(a_{2k}^{(j)} - a_{2k+1}^{(j)}) = (DHa^{(j)})_k$$

and

$$a_k^{(j-1)} = (DLa^{(j)})_k \,.$$

The reconstruction algorithm can also be cast in a similar form, if we define the upsampling operator.

Definition

The upsampling operator U acts on a square-summable sequence $\{x_k\}_{k\in\mathbb{Z}}$ by

$$(Ux)_k = \begin{cases} x_{k/2}, & k \text{ even} \\ 0, & k \text{ odd} \end{cases} .$$

Remark

Let $\widetilde{H}$ and $\widetilde{L}$ be given by

$$\tilde{h} = (\ldots, 0, 0, \ldots, 0, 0, \underbrace{1}_{k=0}, -1, 0, \ldots)$$

and let

$$\tilde{l} = (\ldots, 0, 0, \ldots, 0, 0, \underbrace{1}_{k=0}, 1, 0, \ldots)\,.$$

Thus,

$$a^{(j)} = \widetilde{L} U a^{(j-1)} + \widetilde{H} U b^{(j-1)}\,.$$

Wavelet decomposition and reconstruction with filters

Split detail from lower resolution level:

$$b_k^{(j-1)} = (DHa^{(j)})_k$$

and

$$a_k^{(j-1)} = (DLa^{(j)})_k \,.$$

Fuse lower resolution and detail level:

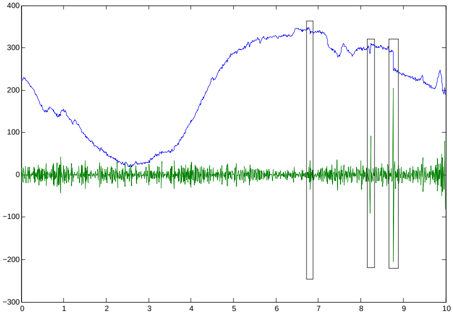
$$a^{(j)} = \widetilde{L}Ua^{(j-1)} + \widetilde{H}Ub^{(j-1)} \,.$$

Application: step detection

Remark

If a function $f \in V_j$ is slowly changing except for finitely many discontinuities, then applying the decomposition shows that only wavelet coefficients $b_k^{(j-1)}$ near discontinuities are large!

Comparison: Signal vs. wavelet coefficients



Compression?

For piecewise constant functions, many coefficients would be exactly zero, i.e. can be discarded. Only need to store detail coefficients close to steps, and coefficients for low resolution level everywhere. Note: Downsampling reduces data by factor 2 in each decomposition step. Compression!

Question

Is there a version of ϕ, ψ which will compress piecewise quadratic/cubic/etc functions similarly?

Answer

The family of Daubechies wavelets have the desired property.

Definition

Let $\{V_j\}$ be a family of subspaces in $L^2(\mathbb{R})$ such that any Cauchy sequence in each V_j converges. Then $\{V_j\}$ is called a **multiresolution analysis** if the following properties hold.

1. $V_j \subset V_{j+1}$ for all $j \in \mathbb{Z}$,
2. $\overline{\cup_j V_j} = L^2(\mathbb{R})$ (union is dense)
3. $\cap_j V_j = \{0\}$
4. $f \in V_j \leftrightarrow f(2^{-j}x) \in V_0$
5. There is $\phi \in V_0$ such that $\{\phi(x-k)\}_{k\in\mathbb{Z}}$ is an orthonormal basis for V_0. Each V_j is called an approximation subspace. The resulting $W_j = V_j^\perp \cap V_{j+1}$ are called detail spaces.

In short, MRAs have decomposition and reconstruction algorithms like the Haar wavelet transform.

Example

As first example of a multiresolution analysis, we note that the Haar scaling function satisfied the required properties.

Example

A second example of a multiresolution analysis is given by the approximation spaces which consist of $2^j\pi$-bandlimited functions:

$$V_j = \{f \in L^2(\mathbb{R}) : \hat{f}(\omega) = 0 \text{ for all } |\omega| > 2^j\pi\}.$$

Remark

The Daubechies construction is “in between” these two examples.

The scaling relation

Theorem

If $\{V_j\}$ *is a multiresolution analysis with scaling function* ϕ*, then there is a sequence* $\{p_k\}_{k\in\mathbb{Z}} \in \ell^2(\mathbb{Z})$ *such that for almost every* $x \in \mathbb{R}$*,*

$$\phi(x) = \sum_{k\in\mathbb{Z}} p_k \phi(2x-k)$$

and

$$p_k = 2\int_{-\infty}^{\infty} \phi(x)\overline{\phi(2x-k)}dx\,.$$

Example

For the Haar MRA, $p_0 = p_1 = 1$, and all other p_j are zero.

Wavelet design in the frequency domain

Problem

Find scaling coefficients $\{p_k\}_{k \in \mathbb{Z}}$ *that belong to MRAs. Design MRAs with special properties such as smooth scaling functions, compactly supported ones, etc.*

Strategy

It will become apparent that the frequency-domain formulation is convenient for such problems.

To this end, we examine the two-scale relation. We use the notation

$$p(\omega) = \frac{1}{2} \sum_{k \in \mathbb{Z}} p_k e^{-ik\omega}$$

or

$$P(z) = \frac{1}{2} \sum_{k \in \mathbb{Z}} p_k z^k .$$

Theorem

If the integer translates of $\phi \in L^2(\mathbb{R})$ form an orthonormal set and $\phi(x) = \sum_{k\in\mathbb{Z}} p_k \phi(2x - k)$, then $P(z)$ satisfies the **quadrature mirror property**

$$|P(z)|^2 + |P(-z)|^2 = 1, \quad |z| = 1\,.$$

Example (Haar MRA)

For

$$P(z) = \frac{1}{2}(1 + z)$$

we obtain for $|z| = 1$ that

$$|P(z)|^2 + |P(-z)|^2 = \frac{1}{4}(|1 + z|^2 + |1 - z|^2) = \frac{1}{4}(2 + 2|z|^2) = 1\,.$$

The next theorem addresses is whether the quadrature-mirror property of $P(z)$ is enough to create a scaling function ϕ for an MRA.

Theorem

Given $P(z) = \frac{1}{2}\sum_k p_k z^k$ *with a summable sequence* $\{p_k\}$ *satisfying*

1. $P(1) = 1$,
2. $|P(z)|^2 + |P(-z)|^2 = 1, |z| = 1$,
3. $|P(e^{it})| > 0, |t| \leq \pi/2$,

then the iteration

$$\phi_n(x) = \sum_k p_k \phi_{n-1}(2x - k)$$

starting with the Haar scaling function ϕ_0 *converges to the scaling function* ϕ *of an MRA.*

Suppose we are given an MRA with summable scaling coefficients $\{p_k\}_{k \in \mathbb{Z}}$, which satisfy the three properties on the preceding slide.

Proposition

A wavelet ψ is obtained by

$$\psi(x) = \sum_k (-1)^k \overline{p_{1-k}} \phi(2x - k)$$

or alternatively

$$\hat{\psi}(\xi) = Q(e^{-i\xi/2}) \hat{\phi}(\xi/2)$$

with

$$Q(z) = \frac{1}{2} \sum_k (-1)^k \overline{p_{1-k}} z^k .$$

If ϕ is integrable, then $|\hat{\phi}(\xi/2)| \le M$ and

$$|\hat{\psi}(\xi)| \le M|Q(e^{-i\xi/2})|\,.$$

Corollary

This means, if $Q(e^{-i\xi/2})$ has vanishing derivatives at $\xi = 0$, then so does $\hat{\psi}$.

For example assuming $\hat{\psi}(0) = \hat{\psi}'(0) = 0$, then we can conclude

$$\int_{-\infty}^{\infty} \psi(x)dx = \int_{-\infty}^{\infty} x\psi(x)dx = 0\,.$$

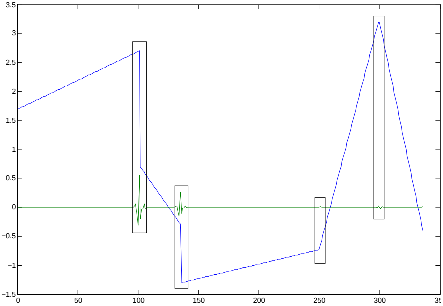
Consequently, any function which is linear on the support of ψ, $f(x) = ax + b$ for all x where $\psi(x) \neq 0$, gives vanishing wavelet coefficients

$$\int_{-\infty}^{\infty} f(x)\psi(x)dx = 0\,.$$

Compression

Wavelet coefficients for a piecewise linear signal

If the wavelet has a vanishing first moment, then the coefficients are zero where the signal is linear:



Construction of the Daubechies wavelet

Try the simplest case first, a polynomial P.

Problem

Find a polynomial P such that

$$p(\xi) = P(e^{-i\xi})$$

has the following properties:

1. $p(0) = 1$,
2. $|p(\xi)|^2 + |p(\xi + \pi)|^2 = 1$,
3. $|p(\xi)| > 0$ *for* $-\pi/2 \leq \xi \leq \pi/2$

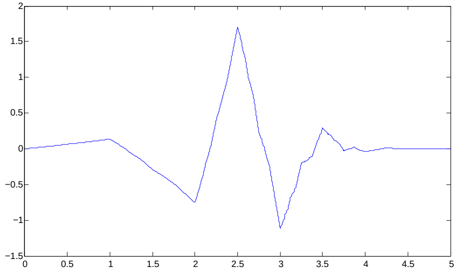
and the associated ψ has vanishing zeroth and first moments.

Example

The polynomial

$$p_3(\xi) = \frac{1+\sqrt{3}}{8} + \frac{3+\sqrt{3}}{8}e^{-i\xi} + \frac{3-\sqrt{3}}{8}e^{-2i\xi} + \frac{1-\sqrt{3}}{8}e^{-3i\xi}$$

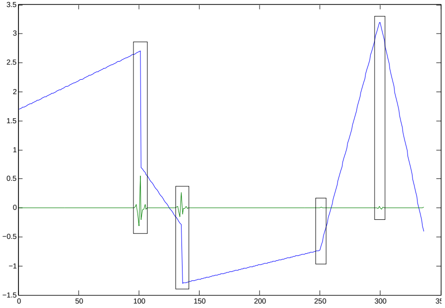
is an example, which belongs to the Daubechies wavelet.

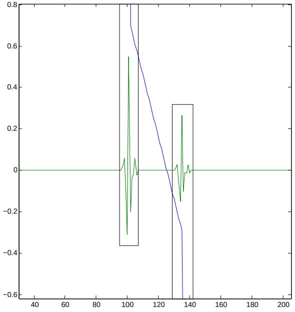


Theorem

The Daubechies wavelet ψ is continuous and has vanishing zeroth and first moments,

$$\int_{-\infty}^{\infty} \psi(x)dx = \int_{-\infty}^{\infty} x\psi(x)dx = 0\,.$$





Outlook

Beyond 60 slides:

- Daubechies wavelets for piecewise quadratic, cubic etc. can also be found.
- More generally, given a space of typical signals, we can find wavelets which mimic signal properties and are most efficient for decomposition, denoising and compression.
- Wavelets in higher dimensions – stay tuned!
- Wavelets and oversampling? Ditto!
- More detail, Fourier and wavelets in smaller portions: MATH 4355 – Mathematics of Signal Representations – Spring 2009.

Literature:

- Boggess and Narcowich, A First Course in Wavelets with Fourier Analysis, Prentice Hall, 2001.
- Mallat, A Wavelet Tour of Signal Processing, Academic Press, 1999.
- Wojtaszczyk, A Mathematical Introduction to Wavelets, Cambridge University Press, 1997.
- Weiss and Hernandez, A First Course on Wavelets, CRC Press, 1996.