High-Dimensional Measures and Geometry

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2.1 High-dimensional Gaussian measures

2.1.1 Proposition. *Let γ_n be the standard Gaussian measure on $\mathbb{R}^n$. Then for any $\delta \geq 0,\ n \in \mathbb{N}$,*

(i) $\gamma_n(\{x \in \mathbb{R}^n : \|x\|^2 \geq n+\delta\}) \leq (\frac{n}{n+\delta})^{-n/2} e^{-\delta/2}$

and if $0 < \delta \leq n$, then

(ii) $\gamma_n(\{x \in \mathbb{R}^n : \|x\|^2 \leq n-\delta\}) \leq (\frac{n}{n-\delta})^{-n/2} e^{-\delta/2}$.

Proof. We first prove (i). Let $0 < \lambda < 1$, then

$$\|x\|^2 \geq n+\delta \Rightarrow \lambda\frac{\|x\|^2}{2} \geq \lambda\frac{(n+\delta)}{2} \Rightarrow e^{\lambda\|x\|^2/2} \geq e^{\lambda(n+\delta)/2}.$$

Then, according to the Laplace transform technique,

$$\begin{aligned}\gamma_n(\{x \in \mathbb{R}^n : \|x\|^2 \geq n+\delta\}) &\leq e^{-\lambda(n+\delta)/2}\int_{\mathbb{R}^n} e^{\lambda\|x\|^2/2} d\gamma_n \\ &= \frac{e^{-\lambda(n+\delta)/2}}{(2\pi)^{n/2}}\int_{\mathbb{R}^n} e^{-(1-\lambda)\|x\|^2/2} dx \\ &= \frac{e^{-\lambda(n+\delta)/2}}{(1-\lambda)^{n/2}}.\end{aligned}$$

Finally, choosing $\lambda = \frac{\delta}{n+\delta}$ gives (i).

To prove (ii), again we consider $\lambda > 0$, then

$$\|x\|^2 \leq n-\delta \Rightarrow e^{-\lambda\|x\|^2/2} \geq e^{-\lambda(n-\delta)/2}.$$

Using the Laplace transform we get,

$$\begin{aligned}\gamma_n(\{x \in \mathbb{R}^n : \|x\|^2 \leq n-\delta\}) &\leq e^{\lambda(n-\delta)/2}\int_{\mathbb{R}^n} e^{-\lambda\|x\|^2/2} d\gamma_n \\ &= \frac{e^{\lambda(n-\delta)/2}}{(2\pi)^{n/2}}\int_{\mathbb{R}^n} e^{-(1+\lambda)\|x\|^2/2} dx \\ &= \frac{e^{\lambda(n-\delta)/2}}{(1+\lambda)^{n/2}}.\end{aligned}$$

Lastly, (ii) follows by choosing $\lambda = \frac{\delta}{n-\delta}$. $\square$

2.1.2 Corollary. *Let γ_n denote the standard Gaussian measure on $\mathbb{R}^n$. Then for $0 < \epsilon < 1$ and $n \in \mathbb{N}$ we have,*

(i) $\gamma_n(\{x \in \mathbb{R}^n : \|x\|^2 \geq \frac{n}{1-\epsilon}\}) \leq e^{-n\epsilon^2/4}$

(ii) $\gamma_n(\{x \in \mathbb{R}^n : \|x\|^2 \leq (1-\epsilon)n\}) \leq e^{-n\epsilon^2/4}$.

Proof. We let $\delta = \frac{n\epsilon}{1-\epsilon}$ in (i) of Proposition 2.1.1, then

$$\begin{aligned}\gamma_n(\{x \in \mathbb{R}^n : \|x\|^2 \geq \frac{n}{1-\epsilon}\}) &\leq (1-\epsilon)^{-n/2} e^{\frac{-n\epsilon}{2(1-\epsilon)}} \\ &= e^{\frac{-n}{2}(ln(1-\epsilon)+\frac{\epsilon}{1-\epsilon})}.\end{aligned}$$

By writing the power series of $ln(1-\epsilon)$ and $\frac{\epsilon}{1-\epsilon}$ we get

$$ln(1-\epsilon) + \frac{\epsilon}{1-\epsilon} \leq e^{\epsilon^2/2}$$

Thus,

$$\gamma_n(\{x \in \mathbb{R}^n : \|x\|^2 \geq \frac{n}{1-\epsilon}\}) \leq e^{\frac{-n\epsilon^2}{4}}.$$

To prove (ii), we take $\delta = n\epsilon$ in (ii) of Proposition 2.1.1. Then,

$$\begin{aligned}\gamma_n(x \in \mathbb{R}^n : \|x\|^2 \leq (1-\epsilon)n) &\leq (1-\epsilon)^{n/2} e^{n\epsilon/2} \\ &= e^{\frac{n}{2}(ln(1-\epsilon)+\epsilon)}.\end{aligned}$$

Again, using the power series of $ln(1-\epsilon)$ we get

$$\gamma_n(x \in \mathbb{R}^n : \|x\|^2 \leq (1-\epsilon)n) \leq e^{\frac{-n\epsilon^2}{4}}.$$

□

Note, in Corollay 2.1.2 if we choose a sequence $\{\epsilon_n\} \to 0$ such that the sequence $\{n{\epsilon_n}^2\} \to \infty$, then $\gamma_n(\{x \in \mathbb{R}^n : n(1-\epsilon_n) \leq \|x\|^2 \leq \frac{n}{1-\epsilon_n}\}) \to 1$. So, we conclude that the Gaussian measure is supported on a thin spherical shell in high dimensions. We use this later to show concentration of measure for γ_n implies concentration for μ, the spherical measure.

2.2 Gaussian measures and projections

let γ_n be the standard Gaussian measure and $V \subseteq \mathbb{R}^n$ a k-dimensional subspace with orthogonal projection $P : \mathbb{R}^n \to V$, then the measure induced by P on V is γ_k (with respect to any orthonormal basis on V).

2.2.1 Question. Why is the induced measure γ_k?

Without loss of generality, we can rotate the sphere so that V is spanned by $\{e_1, \ldots, e_k\}$. Now, if we let $A \subseteq V$ is measurable, then

$$\begin{aligned}\gamma_n(P^{-1}(A)) &= \frac{1}{(2\pi)^{n/2}} \int_{A\times\mathbb{R}^{n-k}} e^{-\|x\|^2/2} dx \\ &= \frac{1}{(2\pi)^k} \int_A e^{-\|x\|^2/2} dx \\ &= \gamma_k(A).\end{aligned}$$

We can now state a lemma by Johnson and Lindenstrauss, Part I.

2.2.2 Lemma. *Let γ_n be the standard Gaussian measure, V a k-dimensional subspace of $\mathbb{R}^n$, P the orthogonal projection onto V, then for $0 < \epsilon < 1$,*

(i) $\gamma_n(\{x \in \mathbb{R}^n : \sqrt{\frac{n}{k}}\|Px\| \geq (1-\epsilon)^{-1}\|x\|\}) \leq e^{-\frac{\epsilon^2 k}{4}} + e^{-\frac{\epsilon^2 n}{4}}$

(ii) $\gamma_n(\{x \in \mathbb{R}^n : \sqrt{\frac{n}{k}}\|Px\|\} \leq (1-\epsilon)\|x\|) \leq e^{-\frac{\epsilon^2 k}{4}} + e^{-\frac{\epsilon^2 n}{4}}$.

Proof. Since the Gaussian measure γ_n is concentrated in a spherical shell, we recall

$$\gamma_n(\{x \in \mathbb{R}^n : \|x\| \geq \sqrt{n(1-\epsilon)}\}) \geq 1 - e^{-\frac{\epsilon^2 n}{4}}.$$

Similarly for γ_k,

$$\gamma_k(\{x \in \mathbb{R}^k : \|x\| \leq \sqrt{\frac{k}{1-\epsilon}}\}) \geq 1 - e^{-\frac{\epsilon^2 k}{4}}.$$

Note that

$$\begin{aligned}\gamma_n(\{x \in \mathbb{R}^n : x \in E_1 \mathrm{or} Px \in E_2\}) &\leq \gamma_n(\{x \in \mathbb{R}^n : x \in E_1\}) + \gamma_n(\{x \in \mathbb{R}^n : Px \in E_2\}) \\ &= \gamma_n(E_1) + \gamma_k(E_2)\end{aligned}$$

Thus,

$$\gamma_n(\{x \in \mathbb{R}^n : \|x\| < \sqrt{n(1-\epsilon)} \text{ or } \|Px\| > \sqrt{n(1-\epsilon)}\}) \leq e^{-\frac{\epsilon^2 n}{4}} + e^{-\frac{\epsilon^2 k}{4}}.$$

and so,

$$\begin{aligned}\gamma_n(\{x \in \mathbb{R}^n : \frac{\|x\|}{\sqrt{n}} \leq \sqrt{1-\epsilon} &\leq (1-\epsilon)\frac{\|Px\|}{\sqrt{k}}\}) \\ &\geq 1 - (e^{-\frac{\epsilon^2 n}{4}} + e^{-\frac{\epsilon^2 k}{4}})\end{aligned}$$

which is the claimed bound for the measure of $\{x \in \mathbb{R}^n : \sqrt{\frac{n}{k}}\|Px\| \geq (1-\epsilon)^{-1}\|x\|\}$. Similarly, we obtain (ii) from (ii) of Corollary 2.1.2. □

2.3 Gaussian vs. surface measures

Define a normalized map $\phi : \mathbb{R}^n \setminus \{0\} \to S^{n-1}, x \mapsto \frac{x}{\|x\|}$, then the image measure of γ_n under ϕ is μ_n, the surface measure of S^{n-1}.

2.3.1 Corollary. *If μ_n is rotation-invariant probability measure on S^{n-1}, $V \subseteq \mathbb{R}^n$ a k-dimensional subspace and P the orthogonal projection onto V, then*

$$\mu_n(\{x \in \mathbb{R}^n : \sqrt{\frac{n}{k}}\|Px\| \geq (1-\epsilon)^{-1}\}) \leq e^{-\frac{\epsilon^2 n}{4}} + e^{-\frac{\epsilon^2 k}{4}}$$

and

$$\mu_n(\{x \in \mathbb{R}^n : \sqrt{\frac{n}{k}}\|Px\| \leq 1-\epsilon\}) \leq e^{-\frac{\epsilon^2 n}{4}} + e^{-\frac{\epsilon^2 k}{4}}.$$

Proof. Changing the norm of x on both sides does not changes the validity of (i) and (ii) in Lemma 2.2.2. □