High-Dimensional Measures and Geometry

Lecture Notes from Jan 26, 2010

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Denote by $G_k(\mathbb{R}^n)$ the Grassmannian, which is the collection of k-dimensional subspaces of $\mathbb{R}^n$. Define a distance on $G_k(\mathbb{R}^n)$ by the operator norm of the difference between corresponding orthogonal projections. That is, $P_1 : \mathbb{R}^n \to V_1$, $P_2 : \mathbb{R}^n \to V_2$, with V_1, V_2 k-dimensional, then $d(V_1, V_2) = ||P1 - P2||$. This distance is invariant under the orthogonal group. So, $||P_1 - P_2|| = ||OP_1O^* - OP_2O^*|| = ||O(P_1 - P_2)O^*||$, $O \in \mathcal{O}(n)$, the set of unitary operators on $\mathbb{R}^n$.

Also, $\mathcal{O}(n)$ acts transitively on projections, for all rank-k P_1, P_2, $\exists O \in \mathcal{O}(n)$ s.t. $P_2 = OP_1O^*$ $\Rightarrow$ $\exists!$ Borel probability measure on $G_k(\mathbb{R}^n)$, invariant under $\mathcal{O}(n)$, we denote this measure by $\mu_{n,k}$.

This measure can be obtained from the left-invariant Haar measure ν_n on $\mathcal{O}(n)$ by the map

$$\Psi : \mathcal{O} \to OP_{V_1}O^*$$

P_{V_1} an orthogonal projection onto some fixed k-dimensional subspace.

In terms of subspaces, we have

$$\mu_{n,k}(V) = \nu_n(\{U \in \mathcal{O}(n) : U(V_1) \in V\}), V \in G_k(\mathbb{R}^n)$$

2.3.1 Question. Why is this identity true?

This is because the image measure is invariant under the action of $\mathcal{O}(n)$, by the commutative diagram below.

$$\begin{array}{ccc} \mathcal{O}(n) & \xrightarrow{O \mapsto O'O \ \ [1]} & \mathcal{O}(n) \\ \downarrow \Psi & & \downarrow \Psi \\ G_k(\mathbb{R}^n) & \xrightarrow{V \mapsto O'V \ \ [1]} & G_k(\mathbb{R}^n) \end{array} \tag{1}$$

[1] This is left multiplication by O', for some fixed $O' \in \mathcal{O}(n)$

The "effective map" between $G_k(\mathbb{R}^n)$ is invariant under $\mathcal{O}(n)$ because $O'P_V(O')^* = O'OP_{V_1}O^*(O')^*$ and this projection has range $O'(O(V_1)) = (O'O)(V_1)$

2.3.2 Lemma. *Let $x \in \mathbb{R}^n, x \neq 0$, let $\mu_{n,k}$ be the $\mathcal{O}(n)$-invariant measure on $G_k(\mathbb{R}^n)$, and for each $V \in G_k(\mathbb{R}^n)$, let P_V denote orthogonal projection onto V. Then, for $0 < \epsilon < 1$,*

$$\mu_{n,k}(\{V \in G_k(\mathbb{R}^n); \sqrt{\frac{n}{k}}||P_V(x)|| \geq \frac{1}{1-\epsilon}||x||\}) \leq \exp(-\epsilon^2 k/4) + \exp(-\epsilon^2 n/4)$$

and

$$\mu_{n,k}(\{V \in G_k(\mathbb{R}^n); \sqrt{\frac{n}{k}}||P_V(x)|| \leq (1-\epsilon)||x||\}) \leq \exp(-\epsilon^2 k/4) + \exp(-\epsilon^2 n/4)$$

Proof. Without loss of generality, choose $||x|| = 1$. Choose any k-dimensional subspace, V_1, and if $U \in \mathcal{O}(n)$, let $V = U(V_1)$, P_V the orthogonal projection onto V_1, and use the fact that the measure ν_n on $\mathcal{O}(n)$ induces the Grassmanian measure $\mu_{n,k}$.

This implies,

$$\mu_{n,k}(\{V \in G_k(\mathbb{R}^n); \sqrt{\frac{n}{k}}||P_V(x)|| \geq \frac{1}{1-\epsilon}\}) = \nu_n(\{U \in \mathcal{O}(n); \sqrt{\frac{n}{k}}||P_{U(V_1)}(x)|| \geq \frac{1}{1-\epsilon}\})$$

and

$$\mu_{n,k}(\{V \in G_k(\mathbb{R}^n); \sqrt{\frac{n}{k}}||P_V(x)|| \leq (1-\epsilon)\}) = \nu_n(\{U \in \mathcal{O}(n); \sqrt{\frac{n}{k}}||P_{U(V_1)}(x)|| \leq (1-\epsilon)\})$$

The projected length of x is

$$||P_{U(V_1)}(x)|| = ||U^* P_{U(V_1)} U U^* x|| = ||P_{V_1} U^* x||$$

and the image measure induced by ν_n under $\Phi_x : \mathcal{O}(n) \to S^{n-1}, U \mapsto U^* x$ is the surface measure on sphere, μ_n.

Thus,

$$\nu_n(\{U \in \mathcal{O}(n); \sqrt{\frac{n}{k}}||P_{U(V_1)}(x)|| \geq \frac{1}{1-\epsilon}\}) = \mu_n(\{y \in S^{n-1}; \sqrt{\frac{n}{k}}||P_{V_1}(y)|| \geq \frac{1}{1-\epsilon}\})$$

and

$$\nu_n(\{U \in \mathcal{O}(n); \sqrt{\frac{n}{k}}||P_{U(V_1)}(x)|| \leq (1-\epsilon)\}) = \mu_n(\{y \in S^{n-1}; \sqrt{\frac{n}{k}}||P_{V_1}(y)|| \leq (1-\epsilon)\})$$

now applying the corollary in section 2.3 (gaussian v.s. surface measure), finishes the proof. □

Summary: Norm reduction for vectors on S^{n-1} under a fixed projection is "mostly" by factor $\sqrt{\frac{k}{n}}(1 \pm \epsilon)$, same is true for fixed vector under projections onto "many subspaces", in $G_k(\mathbb{R}^n)$.

Question: what about more than one vector?

2.3.3 Theorem. *(Johnson-Lindenstrauss, Part II)*
Let $a_1, ..., a_N$ be points in $\mathbb{R}^n$, given $\epsilon > 0$, choose $k \in \mathbb{N}$ $s.t.$

$$N(N-1)(\exp(-k\epsilon^2/4) + \exp(-n\epsilon^2/4)) \leq \frac{1}{3}$$

and let $G_k(\mathbb{R}^n)$ be the set of k-dimensional subspaces, then

$$\mu_{n,k}(\{V \in G_k(\mathbb{R}^n); (1-\epsilon)||a_i - a_j|| \leq \sqrt{\frac{n}{k}}||P_V(a_i - a_j)|| \leq \frac{1}{1-\epsilon}||a_i - a_j|| \ \forall\ 1 \leq i \leq j \leq N\}) \geq \frac{2}{3}$$

Proof. Let $c_{ij} = a_i - a_j, i > j$, we count $\binom{N}{2} = N(N-1)/2$ such differences, and $||P_V c_{ij}|| = ||P_V a_i - P_V a_j||$.

The set of subspaces V for which $\sqrt{\frac{n}{k}}||P_V c_{ij}|| \geq \frac{1}{1-\epsilon}||c_{ij}||$ or $\sqrt{\frac{n}{k}}||P_V c_{ij}|| \leq (1-\epsilon)||c_{ij}||$ for at least one pair $\{i, j\}$, $i \neq j$, is the union of all the subspaces for which one specific c_{ij} either contracts too much or too little under the projection P_V. There are $N(N-1)/2$ such pairs, and for each pair there are two possibilities for the norm bound. This means, we have a set which is the union of $N(N-1)$ subsets, each subset with measure at most $\exp(-k\epsilon^2/4) + \exp(-n\epsilon^2/4)$. Now by our assumption and the union bound over choices $i, j \in \{1, 2, ..., N\}, i \neq j$ the union of these sets has measure at most $\frac{1}{3}$. Now taking the complement gives the desired estimate of the measure.

□

2.3.4 Question. What about infinitely many vectors, i.e. $\mathrm{span}\{a_1, ..., a_T\}$, for some $T \in \mathbb{N}$? See "restricted isometry property".

Need to choose set of points $Q \subset \{x \in \mathrm{span}\{a_1, ..., a_T\}; ||x|| = 1\}$, "sufficiently dense", apply Johnson-Lindenstrauss to Q, combine this with triangle inequality to get estimate for all points.