High-Dimensional Measures and Geometry

Lecture Notes from Feb 18, 2010

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6.2.1 Theorem. *Let $\tilde{S}^{n-1} = \{x \in \mathbb{R}^n : \ \|x\| = \sqrt{n}\}$ and $\tilde{\mu}_n$ be the rotation invariant Borel probability measure on $\tilde{S}^{n-1}$. Consider*

$$\Phi : \tilde{S}^{n-1} \to \mathbb{R} \text{ given by } \Phi(x_1, ..., x_n) = x_1.$$

Let ν_n be the the probability measure on $\mathbb{R}$ induced by Φ. If γ_1 is the standard Gaussian measure on $\mathbb{R}$ then for any Borel set A

$$\lim_{n\to\infty} \nu_n(A) = \gamma_1(A) = \int_A \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$$

and the density of ν_n converges uniformly on compact subsets to γ_1.

Proof. It is enough to show the first part of the claim for an open interval $A = (a, b)$. Assume n is large. Note that $\Phi^{-1}(A) = \{x \in \mathbb{R}^n : \ a < x_1 < b\}$. So for a fixed $a < x_1 < b$, $\Phi^{-1}(\{x_1\})$ is the $n-2$ sphere of radius $\sqrt{n - x_1^2}$ which we denote by $\tilde{S}^{n-2}(\sqrt{n - x_1^2})$. Hence

$$\nu_n(A) = \tilde{\mu}_n(\Phi^{-1}(A)) = \underbrace{\frac{1}{n^{(n-1)/2}|S^{n-1}|}}_{normalization} \int_a^b \underbrace{\left|\tilde{S}^{n-2}\left(\sqrt{n - x_1^2}\right)\right|}_{(\sqrt{n-x_1^2})^{n-2}|\tilde{S}^{n-2}|} \left(1 + (\underbrace{\frac{dr(x_1)}{dx_1}}_{\frac{2x_1}{2\sqrt{n-x_1^2}}})^2\right)^{1/2} dx_1$$

$$= \frac{n^{(n-2)/2}|S^{n-2}|}{n^{(n-1)/2}|S^{n-1}|} \int_a^b \left(1 - \frac{x_1^2}{n}\right)^{\frac{n-2}{n}} \left(1 + \frac{x_1^2}{n - x_1^2}\right)^{\frac{1}{2}} dx_1$$

$$= \frac{1}{\sqrt{\pi n}} \frac{\Gamma(n/2)}{\Gamma(\frac{n-1}{2})} \int_a^b \left(1 - \frac{x_1^2}{n}\right)^{\frac{n-2}{n}} \left(1 + \frac{x_1^2}{n - x_1^2}\right)^{\frac{1}{2}} dx_1$$

$$(by\ MVT) = \frac{1}{\sqrt{\pi n}} \frac{\Gamma(n/2)}{\Gamma(\frac{n-1}{2})} (1 + \eta_n)^{1/2} \int_a^b \left(1 - \frac{x_1^2}{n}\right)^{\frac{n-2}{n}} dx_1$$

for all large enough n for some constant

$$0 \le \eta_n \le \frac{\max(a, b)^2}{n - \max(a, b)^2}$$

so consequently $\lim_{n\to\infty} \eta_n = 0$. Note that the integrand

$$\left(1 - \frac{x_1^2}{n}\right)^{\frac{n-2}{n}} \to e^{-x_1^2/2}$$

not only pointwise but also uniformly over the closed interval $[a, b]$. (Which can be deduced by taking logarithm of both sides and using the inequality

$$\left|\ln\left(1 - \frac{x_1^2}{n}\right) - \frac{x_1^2}{2}\right| \le \frac{C}{n^2}$$

where C depends on a and b.) Cosequently we can use the dominated convergence theorem and the result follows. $\square$

Similirlarly this results holds for projections onto higher dimensional subspaces.

6.2.2 Corollary. *Let $\tilde{S}^{n-1} = \{x \in \mathbb{R}^n : \ \|x\| = \sqrt{n}\}$ and $\tilde{\mu}_n$ be the rotation invariant Borel probability measure on $\tilde{S}^{n-1}$. Consider*

$$\Phi : \tilde{S}^{n-1} \to \mathbb{R}^k \text{ given by } \Phi(x_1, ..., x_n) = (x_1, ..., x_k, 0, ..., 0)$$

Let $\nu_{n,k}$ be the the probability measure on $\mathbb{R}^k$ induced by Φ. If γ_k is the standard Gaussian measure on $\mathbb{R}^k$ then for any Borel set $A \subseteq \mathbb{R}^k$

$$\lim_{n\to\infty} \nu_{n,k}(A) = \gamma_k(A) = \int_A \frac{1}{(2\pi)^{k/2}} e^{-\|x\|^2/2} dx$$

and the density of $\nu_{n,k}$ converges uniformly on compact subsets to density of γ_k.

Proof. We will use the Fourier Transform. Consider

$$F_n(c) = \int_{\mathbb{R}^k} e^{ic\cdot x} d\nu_{n,k}(x) \quad \text{and} \quad G(c) = \int_{\mathbb{R}^k} \frac{1}{(2\pi)^{k/2}} e^{ic\cdot x} e^{-\|x\|^2/2} dx = e^{-\|c\|^2/2}.$$

By definition

$$F_n(c) = \int_{\tilde{S}^{n-1}} e^{ic\cdot x} d\tilde{\mu}_n(x).$$

Since $\tilde{\mu}_n$ is rotation invariant, by a suitable rotation, we may assume that $c = (\|c\|, 0, ..., 0)$. So by using the previous theorem

$$F_n(c) = \int_{\mathbb{R}} e^{ic\cdot x} d\nu_n(x_1)\,.$$

because only the projection of x on the first coordinate enters in $c\cdot x$. This is the Fourier transform of the density for ν_n, which we already know converges uniformly on compact sets in $\mathbb{R}$.

We conclude that $\nu_{n,k}$ has a radial density $\rho_{n,k}$ with $\rho_{n,k}(x) = \rho_n(\|x\|)$ and thus $\rho_{n,k}$ converges uniformly on compacts because any compact set is contained in a ball centered at the origin. $\square$

We are now ready for the main result.

6.2.3 Definition. *A spherical cap centered at $a \in S^{n-1}$ with radius r is*

$$B_a(r) = \{x \in S^{n-1} :\ d(x,a) = \arccos\langle x, a\rangle < r\}.$$

We will use the following theorem of Levy without proof and focus on consequences.

6.2.4 Theorem (Levy). *If $A \subseteq S^{n-1}$, $t > 0$, and $B = B_a(r)$ with $\mu(B) = \mu(A)$ then*

$$\mu(\{x \in S^{n-1} :\ d(x,A) \leq t\}) \geq \mu(\{x \in S^{n-1} :\ d(x,B) \leq t\}) = \mu(B_a(r+t))$$

If B is spherical cap in S^{n-1} with $\mu(B) = 1/2$ then $B = B_a(\pi/2)$ for some a in S^{n-1}. Also, up to a set of measure 0, we have that

$$B_a(\pi/2 + t) = S^{n-1} - B_{-a}(\pi/2 - t)$$

and

$$\mu(B_a(\pi/2 + t)) = 1 - \mu(B_{-a}(\pi/2 - t)) = 1 - \mu(B_a(\pi/2 - t)).$$

So we want an upper bound for $\mu(B_{-a}(\pi/2 - t))$ to find a lower bound for $\mu(B_{-a}(\pi/2 + t))$.