High-Dimensional Measures and Geometry

Lecture Notes from Mar 2, 2010

taken by Nick Maxwell

0.0.1 Theorem. *If* $f : \mathbb{R} \to \mathbb{R}$ *is differentiable with* $E[f] = 0$, *then* $E[f] \leq E\left[\exp(\frac{\pi^2}{8}||\nabla f||^2)\right]$, *assuming this RHS is finite.*

Proof.

$$F(x, y) := f(x) - f(y)$$

$$E\left[e^f\right] \leq E\left[e^F\right] = E\left[\exp\left(-\int_0^{\pi/2} \frac{\partial}{\partial\theta} G(x, y, \theta)\, d\theta\right)\right] = E\left[\exp\left(-\int_0^{\pi/2} \nabla f(x(\theta)) \cdot x'(\theta)\, d\theta\right)\right]$$

with $G(x, y, \theta) = f(x(\theta)) = f(x\cos\theta, y\cos\theta)$, and $x'(\theta) = -x\sin\theta + y\cos\theta = y(\theta)$, which looks like rotation.

$$E\left[e^F\right] = E\left[\exp\left(-\frac{2}{\pi}\int_0^{\pi/2} \frac{\pi}{2}\frac{\partial}{\partial\theta} G(x, y, \theta)\, d\theta\right)\right] \underset{\text{Fubini}}{\overset{\text{Jensen}}{\leq}} \frac{2}{\pi}\int_0^{\pi/2} E\left[\exp\left(-\frac{\pi}{2}\nabla f(x(\theta)) \cdot x'(\theta)\right)\right] d\theta.$$

Fix θ, and change variavles, $\tilde{x}_j = x_j\cos\theta + y_j\sin\theta$, $\tilde{y} = -x_j\sin\theta + y_j\cos\theta$. Then,

$$E[g(x, y)] = E[g(\tilde{x}, \tilde{y})],$$

invariant under rotation. Thus by invartiance of measure,

$$E\left[e^F\right] \leq \frac{2}{\pi}\int_0^{\pi/2} E\left[\exp\left(-\frac{\pi}{2}\nabla f(x) \cdot y\right)\right] d\theta = \frac{2}{\pi}\int_0^{\pi/2} E\left[\exp\left(\frac{\pi^2}{4}||\nabla(x)||^2/2\right)\right] d\theta =$$

$$E\left[\exp\left(\frac{\pi^2}{4}||\nabla(x)||^2/2\right)\right]$$

□

0.0.2 Corollary. *If* $f : \mathbb{R} \to \mathbb{R}$ *is differentiable, with* $||\nabla f|| \leq 1$, *and* $a := E[f]$, *then* $\gamma_n(\{x \in \mathbb{R}^n; |f(x) - a| \leq t\}) \geq 1 - 2e^{-2\frac{t^2}{\pi^2}}$

Proof. Let $\lambda \geq 0$, use Laplace transform method,

$$\gamma_n\left(\{x; f(x) - a \geq t\}\right) \leq e^{-\lambda t} E\left[e^{\lambda(f-a)}\right] \leq e^{-\lambda t} e^{\lambda^2 \pi^2/8},$$

choosing $\lambda = \frac{4t}{\pi^2}$ gives

$$\gamma\left(\{x; |f(x) - a| \leq t\}\right) \geq 1 - 2e^{-\frac{2t^2}{\pi^2 L^2}}$$

$\square$

0.0.3 Question. Can we use result for γ_n, and $\Phi = x \mapsto \frac{x}{||x||}$, to obtain concentration about the mean on spheres?

0.0.4 Idea. Extend differentiable 1-Lipschitz funcitons $f : S^{n-1} \to \mathbb{R}$ to $\mathbb{R}^n$, consider $g(x) = ||x|| f(\frac{x}{||x||})$. Then g is differentiable when f is (except at $x = 0$).

$$D_u g(x) = (D_u ||x||)\, f\left(\frac{x}{||x||}\right) + ||x|| D_u f\left(\frac{x}{||x||}\right)$$

$$|D_u g(x)| \leq (1)||f||_\infty + \max_{||u||=1, u\cdot x=0} ||x|| D_u f\left(\frac{x}{||x||}\right) \leq ||f||_\infty + 1.$$

Now, if $\int f \, d\mu_n = 0$, then $||f||_\infty \leq \frac{\pi}{2}$ because

$$|f(x) - 0| = |f(x) - \int_{S^{n-1}} f(x')\, d\mu_n(x')| = |\int_{s^{n-1}} (f(x) - f(y))\, d\mu_n(y)| \underset{1-\text{lip.}}{\leq} \int_{s^{n-1}} d(x,y)\, d\mu_n(y) = \frac{\pi}{2}$$

As a consequence, if f is differentiable on S^{n-a} and 1-Lipschitx, then g as defined above is differentiable on $\mathbb{R}^n \setminus \{0\}$, and is 3-lipschitz.