High-Dimensional Measures and Geometry

Lecture Notes from March 9, 2010

taken by Anando Sen

6.2.1 Lemma. *Given an n-dimensional vector space W with norm $\|.\|$ and $\Sigma = \{x \in W : \|x\| = 1\}$ then for any $\delta > 0$ there is $S \subseteq \Sigma$ such that,*

1. *For all $x \in S$,*

$$\inf_{x \in S, y \neq x} \|y - x\| < \delta$$

2. *We have,*

$$|S| \leq \left(1 + \frac{2}{\delta}\right)^n$$

Proof. Pick a set S such that $\|y - x\| > \delta$ for any $x, y \in S$ and assume we cannot add any further points to S without violating this bound. This means any point in Σ is at most at distance δ from S which implies (1).
Now, consider $B(y, \frac{\delta}{2})$ for $y \in S$ then $B(x, \frac{\delta}{2}) \cap B(y, \frac{\delta}{2}) = \emptyset$ for $x, y \in S$ and,

$$\begin{aligned}
B\left(x, \frac{\delta}{2}\right) &\subseteq B\left(0, 1 + \frac{\delta}{2}\right) \\
\implies |S| \left|B\left(x, \frac{\delta}{2}\right)\right| &\leq \left|B\left(0, 1 + \frac{\delta}{2}\right)\right| \\
\implies |S| \left(\frac{\delta}{2}\right)^n |B(0,1)| &\leq \left(1 + \frac{\delta}{2}\right)^2 |B(0,1)|
\end{aligned}$$

So we conclude,

$$|S| \leq \left(1 + \frac{2}{\delta}\right)^n$$

which was property (2). □

Now we are ready to prove the Theorem.

6.2.2 Theorem. *There is a* $\kappa > 0$ *such that, if* $\epsilon > 0$ *then for any* n*, any* $1-$*Lipshitz* $f : S^{n-1} \to \mathbb{R}, \exists c$ *(eg. median, average) and a subspace* $V \subseteq \mathbb{R}^n$ *such that,*

$$|f(x) - c| \leq \epsilon \qquad\qquad \forall x \in V \cap S^{n-1}$$

and

$$\dim(V) \geq \frac{\kappa\epsilon^2}{\ln\left(1 + \frac{4}{\epsilon}\right)} n$$

Proof. Pick $k-$dimensional subspace $A \subseteq \mathbb{R}^n$, choose an orthogonal transformation $\mathcal{U} \in \mathcal{O}(n)$ and try $V = \mathcal{U}(A) = \{\mathcal{U}x : x \in A\}$, then we show that a set of $\mathcal{U}$'s of non-zero measure does it. Choose an $\frac{\epsilon}{2}$-net, $S \subset A \cap S^{n-1}$, with

$$|S| \leq \left(1 + \frac{4}{\epsilon}\right)^k = e^{k \ln\left(1+\frac{4}{\epsilon}\right)}$$

and pick $X = \{x \in S^{n-1} : |f(x) - c| \leq \frac{\epsilon}{2}\}$. Then

$$\mu(X) \geq 1 - e^{-\alpha n \epsilon^2}$$

for some $\alpha > 0$, if c is the median or average of f.
If we can find $\mathcal{U}$ such that $\mathcal{U}(S) \subset X$ then on $V = \mathcal{U}(A)$, f is $\epsilon-$close to c.
This is because if $x \in V \cap S^{n-1}$, $\exists y \in \mathcal{U}(S)$ and $\|y - x\| \leq \frac{\epsilon}{2}$, (since $\mathcal{U}(S)$ is an $\frac{\epsilon}{2}-$net for $V \cap S^{n-1}$), $|f(y) - c| \leq \frac{\epsilon}{2}$ from $y \in X$. Also from f being $1-$Lipshitz and traingle inequality $|f(x) - c| \leq \epsilon$.
To find this, choose a "random" $\mathcal{U}$.
Let ν_n be the Haar probability measure on $\mathcal{O}(n)$. Pick $x \in S$. We have $\{\mathcal{U}x : \mathcal{U} \in \mathcal{O}(n)\} = S^{n-1}$. Thus,

$$\nu_n(\{\mathcal{U} : \mathcal{U}x \notin X\}) = \mu_n(S^{n-1} \setminus X) \leq e^{-\alpha n \epsilon^2}$$

and by union bound,

$$\nu_n(\{\mathcal{U} : \mathcal{U}x \notin X \; for \; some \; x \in S\} \leq |S| e^{-\alpha n \epsilon^2} \leq exp\left(k \ln\left(1 + \frac{4}{\epsilon}\right) - \alpha n \epsilon^2\right)$$

Now choosing,

$$k < \alpha \frac{\epsilon^2}{\ln\left(1 + \frac{4}{\epsilon}\right)} n$$

gives upper bound less than one, so desired $\mathcal{U}$ exists. □

6.2.3 Corollary. *Let* P_v *denote the projection onto* $V \in G_k(\mathbb{R}^n)$*, then for any* $\epsilon > 0$*, any subspace* W*,* $\dim W = k$*, there exists* $V \in G_k(\mathbb{R}^n)$ *such that for all* $x \in W$*, we have,*

$$\left| \|Px\| - \sqrt{\frac{k}{n}} \right| < \epsilon$$

where $k \leq \frac{\kappa\epsilon^2}{\ln\left(1+\frac{4}{\epsilon}\right)} n$

(This implies $\left|\|Px\|^2 - \frac{k}{n}\right| \le 2\epsilon$)

Note that rank of P could not be chosen smaller, otherwise restricting to $k-$dimensional subspace would not make it invertible.

6.2.4 Lemma. *Let P_v be the orthogonal projection onto $V \in G_n(\mathbb{R}^n)$ then for any $W = span\{e_{j_1}, e_{j_2}, \dots e_{j_k}\}$ (k basis vectors), $k < n$ and $0 < \delta < \frac{1}{2}$ then,*

$$(*) \qquad (1-\delta)\|x\|\sqrt{\frac{N}{k}}\|P_v x\| \le \frac{1}{1-\delta}\|x\| \qquad \forall x \in V$$

for a set of V's with measure,

$$\mu_{N,n}(\{V : (*) \ holds\}) \ge 1 - 2\left(1 + \frac{8}{\delta}\right)^k e^{-\alpha\left(\frac{\delta}{2}\right)^2 n}$$

Proof. We only have to show $(*)$ for $\|x\| = 1, x \in W$ and $\min_{y\in S}\|x-y\| \le \dfrac{\delta}{4}$ for all $x \in W, \|x\| = 1$. We know there is such an S with,

$$|S| \le \left(1 + \frac{8}{\delta}\right)^k.$$

Now applying J-L lemma we get a set of V's with measure as described, such that,

$$\left(1 - \frac{\delta}{2}\right)\|x\| \le \sqrt{\frac{N}{n}}\|P_v x\| \le \frac{1}{1-\frac{\delta}{2}}\|x\|$$

for all $x \in S$. Now let a be the smallest number such that $\sqrt{\frac{N}{n}}\|P_v x\| \le \frac{1}{1-a}\|x\| \quad \forall x \in W$.

We show $a \le \delta$. To see this, let $x \in W, \|x\| = 1$ and pick $y \in S, \|y - x\| \le \frac{\delta}{4}$.
Then,

$$\sqrt{\frac{N}{n}}\|P_v x\| \le \sqrt{\frac{N}{n}}\|P_v y\| + \sqrt{\frac{N}{n}}\|P_v(x-y)\| \le \frac{1}{1-\frac{\delta}{2}} + \frac{1}{1-a}.\frac{\delta}{4}$$

From the definition of a,

$$\begin{aligned}
& \frac{1}{1-a} \le \frac{1}{1-\frac{\delta}{2}} + \frac{1}{1-a}.\frac{\delta}{4} \\
\implies \quad & \frac{1}{1-a}\left(1 - \frac{\delta}{4}\right) \le \frac{1}{1-\frac{\delta}{2}} \\
\implies \quad & \frac{1}{1-a} \le \frac{1}{1-\frac{\delta}{2}}\frac{1}{1-\frac{\delta}{4}} \le \frac{1}{\left(1-\frac{\delta}{2}\right)^2} \\
\implies \quad & 1 - a \ge \left(1 - \frac{\delta}{2}\right)^2 = 1 - \delta + \frac{\delta^2}{4} \\
\implies \quad & a \le \delta
\end{aligned}$$

From the lower inequality,

$$\begin{aligned}
\sqrt{\frac{N}{n}}\|P_v x\| &\geq \sqrt{\frac{N}{n}}\|P_v y\| - \sqrt{\frac{N}{n}}\|P_v(x-y)\| \geq \frac{1}{1-\frac{\delta}{2}} - \frac{1}{1-a}.\frac{\delta}{4} \\
&\geq 1 - \frac{\delta}{2} - \frac{1}{1-a}.\frac{\delta}{4} \\
&\geq 1 - \frac{\delta}{2} - \frac{1}{\left(1-\frac{\delta}{2}\right)\left(1-\frac{\delta}{4}\right)}.\frac{\delta}{4}
\end{aligned}$$

Now using $0 < \delta < \frac{1}{2}$,

$$\frac{1}{1-\frac{\delta}{2}}.\frac{1}{1-\frac{\delta}{4}} \leq \frac{4}{3}.\frac{8}{7}$$

So,

$$\begin{aligned}
\sqrt{\frac{N}{n}}\|P_v x\| &\geq 1 - \frac{\delta}{2} - \frac{4}{3}.\frac{8}{7}.\frac{\delta}{4} \\
&\geq 1 - \frac{\delta}{2} - \frac{\delta}{3} \\
&\geq 1 - \delta
\end{aligned}$$

□