High-Dimensional Measures and Geometry

Lecture Notes from March 23, 2010

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8 The Isoperimetric Inequality

We will work on the "predual" to concentration, the isoperimetric inequality, which give us sets that grow least when enlarged by ϵ-neighborhood.

8.1 The Prékopa-Leindler Theorem

8.1.1 Theorem. *Let $f, g, h : \mathbb{R}^n \to \mathbb{R}^+$ be integrable functions and assume for some $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$ we have that*

$$h(\alpha x + \beta y) \geq \big(f(x)\big)^\alpha \big(g(y)\big)^\beta.$$

Then

$$\int_{\mathbb{R}^n} h(x)dx \geq \left(\int_{\mathbb{R}^n} f(x)dx\right)^\alpha \left(\int_{\mathbb{R}^n} g(y)dy\right)^\beta$$

Proof. The proof is induction on n. So first consider the case $n = 1$. We omit the trivial cases f or g is 0-a.e. Let c_1 and c_2 be the scalars satisfying

$$c_1 \int_{\mathbb{R}^n} f(x)dx = c_2 \int_{\mathbb{R}^n} g(y)dy = 1.$$

Note that

$$c_1^\alpha c_2^\beta h(\alpha x + \beta y) \geq \big(c_1 f(x)\big)^\alpha \big(c_2 g(y)\big)^\beta.$$

So if we only consider the case that f and g is integrate to 1 then show that the integral of h is greater than or equal to 1 then this will be enough and the general case follows from the upper discussion. So we assume this is the case. (Note that we want to prove a statement about probability density). Consider the distribution functions

$$F(t) = \int_{-\infty}^{t} f(x)dx \quad \text{and} \quad G(t) = \int_{-\infty}^{t} g(x)dx.$$

(Note that F and G are increasing so they have, not necessarily unique, one-sided inverses.) Define $u, v : (0, 1) \to \mathbb{R}$ to be the smallest value satisfying

$$\int_{-\infty}^{u(t)} f(x)dx = t \quad \text{and} \quad \int_{-\infty}^{v(t)} g(x)dx = t.$$

We note that u and v are almost everywhere differentiable functions. By the chain rule we have that

$$u'(t)f(u(t)) = 1 \; a.e. \text{ on } \{t : u'(t) \neq 0\}$$

and

$$v'(t)g(v(t)) = 1 \; a.e. \text{ on } \{t : v'(t) \neq 0\}.$$

Now let

$$w(t) = \alpha u(t) + \beta v(t)$$

Then $w'(t) = \alpha u'(t) + \beta v'(t) \geq \big(u'(t)\big)^{\alpha}\big(v'(t)\big)^{\beta}$ because either u' or v' vanishes or otherwise we have the inequality

$$e^{\ln(\alpha u'(t)+\beta v'(t))} \geq e^{\alpha \ln(u'(t))+\beta \ln(v'(t))}.$$

So we have

$$\int_{-\infty}^{\infty} h(x)dx = \int_0^1 h(\underbrace{w(t)}_{\alpha u(t)+\beta v(t)})w'(t)dt \geq \int_0^1 \big(f(u(t))\big)^{\alpha}\big(g(v(t))\big)^{\beta}\big(u'(t)\big)^{\alpha}\big(v'(t)\big)^{\beta}dt = 1.$$

Hence the proof for $n = 1$ is done. Assume the result is true for $n-1$. Choose the hyperplanes $\{x \in \mathbb{R}^n : x_n = \tau\} \cong \mathbb{R}^{n-1}$. Consider

$$f_1(\tau) = \int_{\mathbb{R}^{n-1}} f(y,\tau)dy, \quad g_1(\tau) = \int_{\mathbb{R}^{n-1}} g(y,\tau)dy \quad \text{and} \quad h_1(\tau) = \int_{\mathbb{R}^{n-1}} h(y,\tau)dy.$$

By the assumptions on h, f and g we have that

$$h(\alpha y_1 + \beta y_2, \alpha\tau_1 + \beta\tau_2) \geq \big(f(y_1,\tau_1)\big)^{\alpha}\big(g(y_2,\tau_2)\big)^{\beta}$$

and if we apply the induction assumption to $h(\cdot, \alpha\tau_1 + \beta\tau_2)$, $f(\cdot,\tau_1)$ and $g(\cdot,\tau_2)$ we get

$$h_1(\alpha\tau_1 + \beta\tau_2) \geq \big(f_1(\tau_1)\big)^{\alpha}\big(g_1(\tau_2)\big)^{\beta}.$$

Now using the induction start with f_1, g_1 and h_1 we have

$$\int_{-\infty}^{\infty} h_1(\tau)d\tau \geq \left(\int_{-\infty}^{\infty} f_1(\tau)d\tau\right)^{\alpha}\left(\int_{-\infty}^{\infty} g_1(\tau)d\tau\right)^{\beta}.$$

By Fubini's Theorem,

$$\int_{-\infty}^{\infty} h_1(\tau)d\tau = \int_{\mathbb{R}^n} h(x)dx, \quad \int_{-\infty}^{\infty} f_1(\tau)d\tau = \int_{\mathbb{R}^n} f(x)dx \quad \text{and} \quad \int_{-\infty}^{\infty} g_1(\tau)d\tau = \int_{\mathbb{R}^n} g(x)dx.$$

So the proof is done. □

As an application of Prékopa-Leindler Theorem we will deduce Brunn-Minkowski inequality.

8.1.2 Definition. *A function $p : \mathbb{R}^n \to \mathbb{R}$ is called log-concave if $p \geq 0$ and*

$$p(\alpha x + \beta y) \geq \big(p(x)\big)^{\alpha}\big(p(y)\big)^{\beta}.$$

for all $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$ and for all $x, y \in \mathbb{R}^n$.

A measure μ is called log-concave if it has a density w.r.t. the the Lebesgue measure which is a log-concave function.

8.1.3 Definition. *Let A and B are subsets of $\mathbb{R}^n$. The set*

$$A + B = \{x + y : \ x \in A, \ \ y \in B\}$$

is called Minkowski sum of A and B.

It is worth mentioning that if A and B are measurable sets then the Minkowski sum $A+B$ need not to be measurable. We can now prove a theorem by Brunn and Minkowski.

8.1.4 Theorem. *Let μ be a log-concave measure on $\mathbb{R}^n$. Let A and B are measurable subsets and $\alpha A + \beta B$ is measurable for every $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$. Then*

$$\mu(\alpha A + \beta B) \geq \big(\mu(A)\big)^{\alpha}\big(\mu(B)\big)^{\beta} \text{ for all } \alpha, \beta \geq 0 \text{ with } \alpha + \beta = 1.$$

Proof. Take the density p of μ w.r.t. Lebesgue measure. Define $f = p\chi_A$, $g = p\chi_B$ and $h = p\chi_{\alpha A + \beta B}$. Then

$$h(\alpha x + \beta y) = p(\alpha x + \beta y)\chi_{\alpha A + \beta B} \geq \big(p(x)\big)^{\alpha}\big(p(y)\big)^{\beta}\chi_A(x)\chi_B(y) = \big(f(x)\big)^{\alpha}\big(g(y)\big)^{\beta}$$

for every $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$ and $x, y \in \mathbb{R}^n$. By observing

$$\int_{\mathbb{R}^n} f(x)dx = \mu(A), \quad \int_{\mathbb{R}^n} g(x)dx = \mu(B) \text{ and } \int_{\mathbb{R}^n} h(x)dx = \mu(\alpha A + \beta B)$$

and applying Prékopa-Leindler we obtain the result. □

8.1.5 Theorem. *Let λ be the Lebesgue measure on $\mathbb{R}^n$ and let A, B and $A+B$ be measurable sets in $\mathbb{R}^n$. Then*

$$\big(\lambda(A+B)\big)^{1/n} \geq \big(\lambda(A)\big)^{1/n} + \big(\lambda(B)\big)^{1/n}.$$

Proof. We skip the trivial case $\mu(A)$ or $\mu(B)$ is 0. Pick

$$\alpha = \frac{\big(\lambda(A)\big)^{1/n}}{\big(\lambda(A)\big)^{1/n} + \big(\lambda(B)\big)^{1/n}} \text{ and } \beta = \frac{\big(\lambda(B)\big)^{1/n}}{\big(\lambda(A)\big)^{1/n} + \big(\lambda(B)\big)^{1/n}}.$$

Then $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$. Let $A_1 = \alpha^{-1}A$ and $B_1 = \beta^{-1}B$. Then

$$\lambda(A_1) = \alpha^{-n}\lambda(A) = (\lambda(A)^{1/n} + \lambda(B)^{1/n})^n$$

and

$$\lambda(B_1) = \beta^{-n}\lambda(B) = (\lambda(A)^{1/n} + \lambda(B)^{1/n})^n.$$

Using Brunn-Minkowski gives

$$\lambda(A+B) = \lambda(\alpha A_1 + \beta B_1) = (\lambda(A_1))^{\alpha}(\lambda(B_1))^{\beta} = (\lambda(A)^{1/n} + \lambda(B)^{1/n})^n.$$

□