High-Dimensional Measures and Geometry

Lecture Notes from March 25, 2010

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8.1.6 Remark. Reversing the steps in the proof of the preceeding corollary shows that if μ is a measure on $\mathbb{R}^n$ satisfying

$$\mu(tA) = t^n \mu(A)$$

for all measurable A, then if $A, B, \alpha A + \beta B$ are measurable for all $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$ and

$$\mu(\alpha A + \beta B) = \left[\mu(\alpha A)^{1/n} + \mu(\beta B)^{1/n}\right]^n,$$

then

$$\mu(\alpha A + \beta B) = \mu(A)^\alpha \mu(B)^\beta.$$

Now, we use this to derive the isoperimetric inequality.

8.1.7 Definition. *For $A = \bar{A} \subset \mathbb{R}^n, \rho \geq 0$, we define*

$$A(\rho) = \{x \in \mathbb{R}^n : d(x, A) \leq \rho\}.$$

We note that $A(\rho) = A + B_\rho$, where $B_\rho = \{x \in \mathbb{R}^n : \|x\| \leq \rho\}$.

8.1.8 Theorem. *For a compact set $A \subset \mathbb{R}^n$ and a closed ball B_r of radius $r \geq 0$ such that $\lambda(A) = \lambda(B_r)$, we have that*

$$\lambda(A(\rho)) \geq \lambda(B_{r+\rho}).$$

Proof. We use the additive form of Brunn-Minkowski inequality

$$\begin{aligned}
\lambda^{\frac{1}{n}}(A(\rho)) &= \lambda^{\frac{1}{n}}(A + B_\rho) \\
&\geq \lambda^{\frac{1}{n}}(A) + \lambda^{\frac{1}{n}}(B_\rho) \\
&= \lambda^{\frac{1}{n}}(B_r) + \lambda^{\frac{1}{n}}(B_\rho) \\
&= r\lambda^{\frac{1}{n}}(B_1) + \rho\lambda^{\frac{1}{n}}(B_1) \\
&= (r + \rho)\lambda^{\frac{1}{n}}(B_1) \\
&= \lambda^{\frac{1}{n}}(B_{r+\rho})
\end{aligned}$$

The last two inequalities stem from homogeneity $\lambda(B_r) = r^n \lambda(B_1)$. □

8.1.9 Remark. For "nice" $A, \lim_{\rho \downarrow 0} \dfrac{\lambda(A(\rho)) - \lambda(A)}{\rho}$ is the "surface area". So the ball has the smallest surface area for given volume.

9 Concetration on the Sphere and on Strictly Convex Surfaces

9.0.1 Definition. *Assume $K \subset \mathbb{R}^n$ is compact, convex and $0 \in K$, then we say that K is strictly convex if for any $\epsilon > 0$, there exsits $\delta > 0$ such that for all $x, y \in \partial K, d(x,y) \geq \epsilon$, then $\frac{x+y}{2} \in (1-\delta)K$. We call this function $\epsilon \mapsto \delta(\epsilon)$ a modulus of convexity.*

9.0.2 Example. The ball $B_1 = \{x \in \mathbb{R}^n : \|x\| \leq 1\}$ is strictly convex. For any two points $x, y \in \partial B$ with $d(x,y) \geq \epsilon$, we have

$$\left\|\frac{x+y}{2}\right\|^2 + \left\|\frac{x-y}{2}\right\|^2 = 2\left(\left\|\frac{x}{2}\right\|^2 + \left\|\frac{y}{2}\right\|^2\right) = 1$$
$$\Rightarrow \left\|\frac{x+y}{2}\right\| = \sqrt{1 - \left\|\frac{x-y}{2}\right\|^2}$$

As $\left\|\frac{x-y}{2}\right\|^2 \leq \frac{\epsilon^2}{4}$, we can choose $1 - \delta = \sqrt{1 - \frac{\epsilon^2}{4}} \leq 1 - \frac{\epsilon^2}{8}$ or even larger with $\delta = \frac{\epsilon^2}{8}$.

Now, we induce a measure μ on the surface ∂K by "projecting". For $A \subset \partial K$, we say that A is measurable if the cone segment

$$cs(A) = \{\alpha x : x \in A, 0 \leq \alpha \leq 1\}$$

is Lebesgue measurable and we define

$$\mu(A) = \frac{\lambda(cs(A))}{\lambda(cs(K))}.$$

9.0.3 Example. $K = B_1$, then μ is rotationally invariant and thus μ is the usual normalized measure on S^{n-1}.

9.0.4 Remark. In general, the measure μ is not the induced "surface" measure. We claim that sets of measure $\frac{1}{2}$ have neighbourhoods of almost full measure. Here, the distance of neighbours is inherited from $\mathbb{R}^n$.

9.0.5 Lemma. *If A is convex and $\alpha, \beta \geq 0$, then $(\alpha+\beta)A = \alpha A + \beta B$.*

Proof. We note that $(\alpha+\beta)A \subset \alpha A + \beta B$ from the definition of Minkowski sum.
Conversely, given $x, y \in A$, then

$$\alpha x + \beta y = (\alpha+\beta)\frac{\alpha}{(\alpha+\beta)}x + (\alpha+\beta)\frac{\beta}{(\alpha+\beta)}y$$
$$= (\alpha+\beta)\left(\frac{\alpha}{(\alpha+\beta)}x + \frac{\beta}{(\alpha+\beta)}y\right)$$

As $\frac{\alpha}{(\alpha+\beta)}x + \frac{\beta}{(\alpha+\beta)}y \in A$ (using convexity), we conclude that $\alpha x + \beta y \in (\alpha+\beta)A$. □

9.0.6 Theorem. *Let $K = \bar{K}$ convex, compact, $0 \in K^\circ$ with a modulus of convixity δ. Let $S = \delta K$ and $A \subset S$ be measurable such that $\mu(A) \geq \frac{1}{2}$, then for any $\epsilon > 0$ with $\delta(\epsilon) \leq \frac{1}{2}$,*

$$\mu\left(\{x \in S : d(x, A) \geq \epsilon\}\right) \leq 2\left(1 - \delta(\epsilon)\right)^{2n} \leq 2e^{-2n\delta(\epsilon)}.$$

Proof. Let $B = \{x \in S : d(x, A) \geq \epsilon\}$. We recall that $x \in A, y \in B$, then we have $\dfrac{x+y}{2} \in (1-\delta)K$. More generally, if $x' \in cs(A), y' \in cs(B)$, the same is true for $\dfrac{x'+y'}{2}$. Because $x' = \alpha x, y' = \beta y$ for $x \in A, y \in B, 0 \leq \alpha, \beta \leq 1$ and so

$$\frac{x'+y'}{2} = \frac{\alpha x + \beta y}{2}.$$

Assume that $\alpha \geq \beta$ and $\alpha > 0$, then $\gamma = \frac{\beta}{\alpha} \leq 1$ and

$$\begin{aligned}
\frac{x'+y'}{2} &= \alpha \frac{x + \gamma y}{2} \\
&= \alpha\left(\gamma \frac{x+y}{2} + (1-\gamma)\frac{x}{2}\right) \\
&= \alpha\gamma \frac{x+y}{2} + \alpha(1-\gamma)\frac{x}{2}
\end{aligned}$$

So $\dfrac{x'+y'}{2} \in \alpha\gamma(1-\delta)K + \alpha(1-\gamma)(1-\delta)K = \alpha(1-\delta)K \subset (1-\delta)K$ (using the above Lemma). This implies that

$$\frac{1}{2}cs(A) + \frac{1}{2}cs(B) \subset (1-\delta)K$$

and thus

$$\lambda\left(\frac{1}{2}cs(A) + \frac{1}{2}cs(B)\right) \leq (1-\delta)^n \lambda(K).$$

Using Brunn-Minkowski inequality, we obtain

$$\lambda\left(\frac{1}{2}cs(A) + \frac{1}{2}cs(B)\right) \geq \left[\lambda(cs(A))\right]^{\frac{1}{2}} \left[\lambda(cs(B))\right]^{\frac{1}{2}}$$

and

$$\begin{aligned}
\mu(B) &= \frac{\lambda(cs(B))}{\lambda(K)} \\
&\leq (1-\delta)^{2n} \frac{\lambda(K)}{\lambda(cs(A))} \\
&= (1-\delta)^{2n} \frac{1}{\mu(A)} \\
&\leq 2(1-\delta)^{2n} \leq 2e^{-2n\delta}.
\end{aligned}$$

□

In particular, if $K = B_1$, then

$$\mu\left(\{x \in S : d(x, A) \geq \epsilon\}\right) \leq 2e^{-\epsilon^2 n/4}.$$