High-Dimensional Measures and Geometry

Lecture Notes from April 6, 2010

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9.5.1 Remark. We can simillarly show that sor $r > s > 0$, we get,

$$\left(\int p^r d\mu \right)^{\frac{q}{r}} \leq c(r,s) \int p^s d\mu.$$

So reverse Holder inequality,

$$\|p\|_r \leq c'(r,s)\|p\|_s$$

is valid also when $r, s < 1$.

We examine the limiting case as $r = 1$ and $s \to 0$. From equality at $s = 0$, we could take endpoint derevative at $s = 0$ to get,

$$\ln \|p\|_1 \leq c + \| \ln p\|_1$$

for integrable $\ln p$.
Without c we know that,

$$\ln \left(\int p d\mu \right) \geq \int \ln p d\mu$$

Key to reverse Holder inequality was bounding measure outside of balls. Now it is essential to bound measure for small balls. We prepare with a lemma.

9.5.2 Lemma. *Given a norm $p : \mathbb{R}^n \to \mathbb{R}$, let $B = \{x \in \mathbb{R}^n : p(x) \leq 1\}$. Let $\delta > 0, \rho > 1$ such that $\mu(\rho B) \geq (1 + \delta)\mu(B)$ then for some $c > 0$ we have,*

$$\mu(tB) \leq ct\mu(B)$$

for all $0 < t < 1$.

Proof. By monotonicity of $t \to \mu(tB)$ it is enough to show this for $t = \frac{1}{2m}, m \in \mathbb{N}$ at the cost of enlarging c.

Choose $m \in \mathbb{N}$ and let $\kappa(m)$ be given by,

$$\frac{\mu\left(\frac{1}{2m}B\right)}{\mu(B)} = \frac{\kappa(m)}{m}$$

then what we want to show is that $\kappa(m)$ is bounded above by some constant depending on ρ and δ.
Idea: Slice ball into "onion rings" with a "core" of radius $\frac{1}{2m}$. We prove that measure of rings is large, so by fixed total measure of the ball, core had small measure.

Enlarge κ is necessary to get $\frac{\mu\left(\frac{1}{2m}B\right)}{\mu(B)} \leq \frac{\kappa(m)}{m}$ and $\kappa(m) \geq \frac{2\delta}{\rho}$. By assumption on ρ and δ, $\mu(\rho B) \geq (1+\delta)\mu(B)$, i.e.

$$\mu(\rho B \setminus B) \geq \delta\mu(B)$$

Define for $\tau \geq 0$,

$$A_\tau = \{x \in \mathbb{R}^n : \tau - \frac{1}{2m} \leq p(x) \leq \tau + \frac{1}{2m}\}$$

Since $[1, \delta)$ can be covered by a disjoint union of at most ρm intervals of length $\frac{1}{m}$, there is $\tau' \geq 1$ such that we get "average" contribution to $\mu(\rho B \setminus B)$,

$$(*) \qquad \mu(A_\tau) \geq \frac{\delta}{\rho m}\mu(B)$$

However taking convex combinations with $0 < \lambda < 1$ gives,

$$\underbrace{\lambda A_\tau}_{\{x:\lambda(\tau-\frac{1}{2m})\leq p(x)\leq\lambda(\tau+\frac{1}{2m})\}} + \underbrace{\frac{(1-\lambda)}{2m}B}_{\{y:p(y)\leq\frac{1-\lambda}{2m}\}} \subseteq A_{\lambda\tau}$$

Now using Brunn's Minkowski inequality,

$$\mu(A_{\lambda\tau}) \geq \mu^\lambda(A_\tau)\mu^{1-\lambda}(\frac{1}{2m}B)$$

with $\lambda = \frac{1}{\tau}, \tau = \tau'$.

□