High-Dimensional Measures and Geometry

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10 High Dimensional Measures and Geometry on Graphs

Let $G = (V, E)$ be a graph without loops or multiple edges. Define distance between two vertices $v, w \in V$ by

$$d(v,w) = \begin{cases} \text{smallest number of edges in a path connecting } v \text{ and } w, & \text{if there is such a path} \\ \infty, & \text{otherwise.} \end{cases}$$

10.0.1 Definition. *The Laplacian Δ on a graph $G = (V, E)$ is the matrix $(\Delta)_{v,w \in V}$ with entries*

$$\Delta_{v,w} = \begin{cases} deg(v) & \text{if } v = w \\ -1 & \text{if } \{v, w\} \in E \\ 0 & \text{otherwise.} \end{cases}$$

After choosing the canonical basis, we can think of Δ as an operator on $\mathbb{R}^V = \{f : V \to \mathbb{R}\}$. If $\mathbb{R}^V$ is equipped with the inner product

$$\langle f, g \rangle = \sum_{v \in V} f(v)g(v),$$

then Δ is Hermitian(Symmetric).

10.0.2 Lemma. *Let $G = (V, E)$ be a graph and let E be oriented such that either vertex of $e \in E$ is chosen as the beginning, e_+ and the other as the end, e_- of the edge. Define L on $E \times V$ by*

$$L_{e,w} = \begin{cases} 1 & \text{if } w = e_+ \\ -1 & \text{if } w = e_- \\ 0 & \text{otherwise} \end{cases}$$

then $\Delta = L^* L$.

Proof. The (v, w) entry of Δ is claimed to be

$$\Delta_{v,w} = \sum_{e \in E} (L^*)_{v,e} L_{e,w}$$

with $(L^*)_{v,e} = L_{e,v}$.
If $v = w$, then we have

$$\begin{aligned}\Delta_{v,v} &= deg(v)\\ &= |\{e \in E : \text{ either } e_+ = v \text{ or } e_- = v\}|\\ &= |\{e \in E : L_{e,v} \in \{+1, -1\}\}|\\ &= \sum_{e \in E} |L_{e,v}|^2.\end{aligned}$$

If $v \neq w$, then there is at most one $e = e'$ such that either $e'_+ = v, e'_- = w$ or $e'_+ = w, e'_- = v$, and if there is no such e', then $\Delta_{v,w} = 0$. Also, $L_{e,v} = L_{e,w} = 0$ for all $e \in E$.
If there is on such e', then $\Delta_{v,w} = -1$, but also

$$\sum_{e \in E} L^*_{v,e} L_{e,w} = L_{e',v} L_{e',w} = -1.$$

This completes the proof of the lemma. □

10.0.3 Corollary. *The Laplacian Δ of $G = (V, E)$ is positive semi-definite. If G is connected, then the eigenspace of Δ with eigenvalue $\lambda = 0$ consists of the constant functions on V.*

Proof. By $\Delta = L^*L$, we have

$$\langle \Delta f, f \rangle = \langle Lf, Lf \rangle = \|Lf\|^2 \geq 0$$

for all $f \in \mathbb{R}^V$. From $\sum_{w \in V} \Delta_{v,w} = 0$, it is clear that if f is constant, then $\Delta f = 0$.
Conversely, assume that $\Delta f = 0$. Pick $v, w \in V$ and choose an orientation such that v, w are connected by a directed path from v to w. Now, we have

$$\begin{aligned}\Delta f = 0 &\Rightarrow \langle \Delta f, f \rangle = 0\\ &\Rightarrow \|Lf\|^2 = 0\\ &\Rightarrow Lf = 0.\end{aligned}$$

Thus, $(Lf)(e) = 0$ for all $e \in E$, which implies

$$\sum_{v \in V} L_{e,v} f_v = f(e_+) - f(e_-) = 0$$

for any e on the path. Therefore, $f(v) = constant$ for all $v \in V$(by connectivity of G). □

From now on, we focus on concentration. An essential is the smallest non-zero eigenvalue, λ_1 of Δ. We write the spectral decomposition of Δ as

$$\Delta = \sum_{j=1}^{m} \lambda_j P_j,$$

$P_j = P_j^* P_j, P_j P_k = 0$ if $j \neq k$.
Note that if $\sum_{v \in V} f_v = 0$, then

$$\begin{aligned}
\langle \Delta f, f \rangle &= \sum_{j=1}^{m} \lambda_j \langle P_j f, f \rangle \\
&\geq \lambda_1 \sum_{j=1}^{m} \langle P_j f, f \rangle \\
&= \lambda_1 \|f\|^2.
\end{aligned}$$

Now, we relate λ_1 to the number of edges between sets.

10.0.4 Theorem. *Let $G = (V, E)$ be a connected graph and $X \subset V$ be a subset of vertices. Let $E(X, V \setminus X)$ be the set of edges connecting X with its complement. Then,*

$$|E(X, V \setminus X)| \geq \lambda_1 \frac{|X||V \setminus X|}{|V|}$$

with λ_1, the smallest non-zero eigenvalue of Δ on G.

Proof. Let χ_X be the indicator function of X. Consider $p = \frac{|X|}{|V|}$, and let $f = \chi_X - p$, then we have

$$\sum_{v \in V} f_v = |X| - p|V| = 0$$

So f is orthogonal to constants. Also note that

$$\begin{aligned}
\|f\|^2 &= \langle \chi_X - p, \chi_X - p \rangle \\
&= |X| - 2\langle \chi_X, p1 \rangle + p^2 |V| \\
&= (1 - p)|X| \\
&= \frac{|V \setminus X|}{|V|} |X|.
\end{aligned}$$

Thus, we have

$$\langle \Delta f, f \rangle \geq \lambda_1 \|f\|^2 = \lambda_1 \frac{|V \setminus X|}{|V|} |X|.$$

On the other hand,

$$\Delta f = \Delta(\chi_X - p) = \Delta(\chi_X) \text{ as } \Delta p = 0.$$

Therefore,

$$\begin{aligned}
\langle \Delta f, f \rangle &= \langle \Delta \chi_X, (\chi_X - p) \rangle \\
&= \langle \chi_X, \Delta(\chi_X - p) \rangle \\
&= \langle \chi_X, \Delta \chi_X \rangle \\
&= \langle \Delta \chi_X, \chi_X \rangle \\
&= \|L\chi_X\|^2 \\
&= \sum_{e \in E} |L\chi_X(e)|^2 \\
&= \sum_{e \in E} |\chi_X(e_+) - \chi_X(e_-)|^2 \\
&= |\{e \in E : e_+ \in X, e_- \notin X \text{ or } e_+ \notin X, e_- \in X\}| \\
&= |E(X, V \setminus X)|.
\end{aligned}$$

Hence, we have

$$|E(X, V \setminus X)| \geq \lambda_1 \frac{|V \setminus X|}{|V|} |X|.$$

□