Practice Exam #1
Math 1450

Fall 2014
Section ____________

Name: ______________________________ **Student ID:** ______________________

1. Find the limit

$$\lim_{h \to 0} \frac{(1+h)^{100} - 1}{h} \quad .$$

2. Find $\frac{dy}{d\theta}$ where

$$y(\theta) = \sec(\theta + \cot\theta)$$

3. Apply the squeeze theorem to derive that the function g defined by $g(x) = x^2 \cos(\frac{1}{x})$ if $x \neq 0$ and $g(0) = 0$ has the derivative $g'(0) = 0$.

4. Find the equation of the tangent line to

$$x^2 + 4xy + y^2 = 13$$

at the point with coordinates $x = 2$ and $y = 1$.

5. A boat leaves a dock and travels south at a speed of 25 km/h. Another boat has been heading east at 20 km/h and reaches the dock one hour after the other boat left there.

   (a) How many hours after the departure of the first boat do both boats have the same distance to the dock?

(b) What is the rate at which their distance changes at that time?

6. Prove, using ϵ and δ that

$$\lim_{h \to 0} \frac{(1+h)^2 - 1}{h} = 2$$

7. A particle is moving along the ellipse $\frac{x^2}{4} + \frac{y^2}{9} = 1$.

(a) Sketch the ellipse, indicating the x and y intercepts i.e. where is crosses the x and y axes.

(b) As the particle passes through the point $(1, \frac{\sqrt{27}}{2})$ its x-coordinate increases at a rate of 1 cm/s. (i) Is the particle going clockwise or anticlockwise around the ellipse?

(ii) How fast is its y-coordinate changing at that time?

(iii) How fast is its distance from the origin changing at that time?

8. Suppose that an object of mass is dropped vertically downwards from a cliff ledge $h_0 = 1125m$ above the ground and falls to earth (neglect air resistance) so its height is

$$h(t) = h_0 - \frac{g}{2}t^2$$

where g is taken as $10ms^{-2}$.

With what velocity does it hit the ground?

9. For the function

$$f(x) = \frac{1}{1+x^2}$$

find the set of x values for which:

(a) f is increasing

(b) f is decreasing

(c) the graph of f is concave up

(d) the graph of f is concave down.

10. Find the absolute maximum, absolute minimum, all local extremal values and all critical points of

$$f(x) = \frac{x+1}{x^2+1}$$

on the closed interval $[-1, \frac{1}{2}]$.

11. Show, using the mean value theorem, or otherwise that the function $f(x) = 4x - 1 - \sin(x)$ has exactly one real root.

12. Find two positive integers such that the sum of the first number and twice the second number is 500 and the product of the numbers is as large as possible.

Math 1450
Practice Exam

1. Find $\lim_{h\to 0} \frac{(1+h)^{100}-1}{h} = f'(1)$ for

$f(x) = x^{100} \quad \Rightarrow \quad f'(1) = 100(1)^{99} = 100$

2. $y(\theta) = \sec(\theta + \cot\theta)$

$y'(\theta) = \sec(\theta+\cot\theta)\tan(\theta+\cot\theta)$
$* (1 - \csc^2\theta)$

3. Want to show if

$$u(h) = \frac{h^2\cos(\frac{1}{h}) - 0}{h} = h\cos(\tfrac{1}{h})$$

then $\lim_{h\to 0} u(h) = g'(0)$.

We know $-1 \leq \cos(\frac{1}{h}) \leq 1$, so

if $f(h) = -|h|$, $v(h) = |h|$

then

$f(h) \leq h\cos(\frac{1}{h}) \leq v(h)$.

By $\lim_{h\to 0} f(h) = \lim_{h\to 0} -|h| = 0$,

$\lim_{h\to 0} v(h) = \lim_{h\to 0} |h| = 0$

and squeeze theorem, also $\lim_{h\to 0} u(h) = g'(0)$
$= 0$.

4. Implicit differentiation

$$2x + 4y + 4xy' + 2yy' = 0$$

$$(4x + 2y)\, y' = -2x - 4y$$

$$y' = -\frac{2x + 4y}{4x + 2y}$$

at $x = 2,\ y = 1$

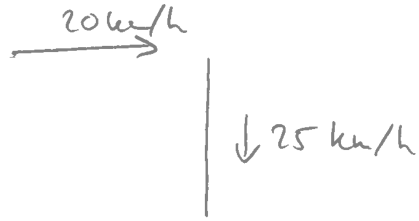
$$y' = -\frac{4 + 4}{8 + 2} = -\frac{4}{5}$$

Tangent line

$$y = -\frac{4}{5}(x - 2) + 1$$

$$= -\frac{4}{5}x + \frac{13}{5}$$

5. a)



$x_1 = 0, \quad y_1 = -25t$

$x_2 = -20 + 20t \quad y_2 = 0$

Same dist. if $|y_1| = |x_2|$

$$20 - 20t = 25t$$

$$20 = 45t$$

$$t = \frac{4}{9}$$

b) distance from each other

$$D = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

$$\frac{dD}{dt} = \frac{1}{2D}\left(2(x_1 - x_2)\frac{d}{dt}(x_1 - x_2) + 2(y_1 - y_2)\frac{d}{dt}(y_1 - y_2)\right)$$

at $t = \frac{4}{9}$, $\quad y_1 = -\frac{100}{9} \quad x_2 = -\frac{100}{9}$

$$D = \sqrt{\frac{100^2}{81} + \frac{100^2}{81}} = \sqrt{2}\,\frac{100}{9}$$

So, rate of change

$$\frac{dD}{dt} = \frac{\cancel{9}\,1}{2\sqrt{2}\,(\cancel{100})}\left(-\frac{2\cancel{00}}{\cancel{9}}(20) + \left(-\frac{2\cancel{00}}{\cancel{9}}\right)(-25)\right)$$

$$= \frac{5}{\sqrt{2}} = \frac{5}{2}\sqrt{2}.$$

6. Let $\varepsilon > 0$.

Want for all $0 < |h| < \delta$ that

$$\underbrace{\left|\frac{(1+h)^2-1}{h} - 2\right|}_{} < \varepsilon$$

$$\left|\frac{\cancel{1} + 2h + h^2 - \cancel{1}}{h} - 2\right|$$

$$|\cancel{2} + h - \cancel{2}| < \varepsilon$$

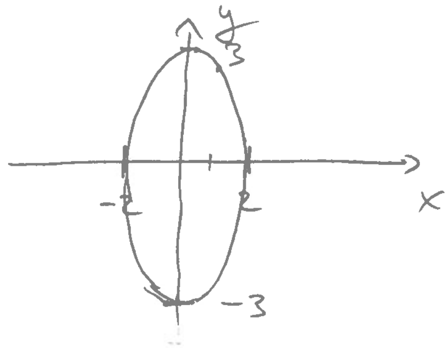
Choose $\delta = \varepsilon$, then if $|h| < \delta = \varepsilon$, we have $|(2+h) - 2| < \varepsilon$, which implies

$$\left|\frac{1+2h+h^2-1}{h} - 2\right| = \left|\frac{(1+h)^2-1}{h} - 2\right| < \varepsilon,$$

as required.

7. $\frac{x^2}{4} + \frac{y^2}{9} = 1.$

a)



b) (i) clockwise

(ii) implicit diff

$$\frac{x}{2}\frac{dx}{dt} + \frac{2y}{9}\frac{dy}{dt} = 0$$

$$\frac{dy}{dt} = -\frac{9x}{4y}\frac{dx}{dt}$$

at $x = 1, \quad y = \frac{\sqrt{27}}{2},$

$$\frac{dy}{dt} = -\frac{9}{4}\frac{2}{\sqrt{27}}(1)$$

$$= -\frac{9}{2}\frac{\sqrt{27}}{27} = -\frac{\sqrt{27}}{6}.$$

(iii) $$\frac{dD}{dt} = \frac{1}{\cancel{2}D}\left(\cancel{2}x\frac{dx}{dt} + \cancel{2}y\frac{dy}{dt}\right)$$

$$= \frac{1}{\sqrt{\underbrace{1+\frac{27}{4}}_{\frac{31}{4}}}}\left(\underbrace{\underbrace{(1)(1) + \frac{\sqrt{27}}{2}\left(-\frac{\sqrt{27}}{6}\right)}_{1-\frac{27}{12}}}_{-\frac{15}{12}}\right)$$

$$= \frac{2}{\sqrt{31}}\left(-\frac{15}{12}\right) = -\frac{5\sqrt{31}}{124}.$$

8. $$h(t) = h_0 - \frac{g}{2}t^2$$

$$h(t_1) = 0 \quad \Rightarrow \quad 0 = h_0 - \frac{g}{2}t_1^2$$

$$t_1 = \sqrt{\frac{2h_0}{g}}$$

$$v(t) = -gt$$

$$v(t_1) = -g\sqrt{\frac{2h_0}{g}} = -\sqrt{2gh_0}$$

$$= -\sqrt{2(10)1125}\ \frac{m}{s}$$

$$= -\sqrt{22500}\ \frac{m}{s}$$

$$= -150\ \frac{m}{s}$$

9. a) $f'(x) = -\frac{1}{(1+x^2)^2}(2x) > 0$

$\iff \quad -x > 0$

$\iff \quad x < 0$ increasing f

b) $f'(x) < 0 \iff x > 0$ decreasing f

c) $f''(x) = -\frac{2}{(1+x^2)^2} + 2\frac{(2x)^2}{(1+x^2)^3}$

$= \frac{8x^2 - 2(1+x^2)}{(1+x^2)^3}$

$= \frac{6x^2 - 2}{(1+x^2)^3} > 0$

$\cancel{6}x^2 \cancel{-2} > \cancel{0} \quad \frac{2}{6} = \frac{1}{3}$

if $x < -\frac{1}{\sqrt{3}}$ or $x > \frac{1}{\sqrt{3}}$

then concave up

d) $f''(x) = \frac{6x^2-2}{(1+x^2)^3} < 0$

$\cancel{6}x^2 \cancel{-2} < \frac{1}{3}$

if $-\sqrt{\frac{1}{3}} < x < \sqrt{\frac{1}{3}}$ then concave down

10. Logarithm trick: (works if $f(x) > 0$ in interval

$$g(x) = \ln f(x) = \ln(x+1) - \ln(x^2+1)$$

$$g'(x) = \frac{1}{x+1} - \frac{2x}{x^2+1} = 0$$

$$\frac{1}{x+1} = \frac{2x}{x^2+1}$$

$$x^2+1 = 2x(x+1)$$

$$= 2x^2 + 2x$$

$$x^2 + 2x - 1 = 0$$

$$x_{1,2} = -1 \pm \sqrt{1+1}$$

$$= -1 \pm \sqrt{2}$$

so crit. point in $[-1, \frac{1}{2}]$ is

$x_c = -1 + \sqrt{2}$

value $f(-1+\sqrt{2}) = \frac{\cancel{-1} + \sqrt{2} + \cancel{1}}{(1 - 2\sqrt{2} + 2) + 1} = \frac{\sqrt{2}}{4 - 2\sqrt{2}}$

endpt. values $f(-1) = 0$

$$f(\tfrac{1}{2}) = \frac{3/2}{1 + \frac{1}{4}} = \frac{1.5}{1.25} < \frac{\sqrt{2}}{4-2\sqrt{2}}$$

$f(-1+\sqrt{2}) = \frac{\sqrt{2}}{4-2\sqrt{2}}$ is local and global max,

$f(-1) = 0$ is global min.

11. We use the intermed. value theorem with continuity of f and

$$f(-\pi) = 4(-\pi) - 1 - \underbrace{\sin(-\pi)}_{0}$$

$$= -4\pi - 1 < 0$$

and

$$f(\pi) = 4\pi - 1 - \cancel{\sin(\pi)} > 0$$

to see f has at least one root.

Next, assume there are c_1, c_2, $c_1 \neq c_2$ s.th. $f(c_1) = f(c_2) = 0$.

By Rolle, there is c_3 in (c_1, c_2) s.th. $f'(c_3) = 0$.

But, $f'(c_3) = 4 - \cos(c_3) \geq 3 > 0$ so this can never happen.

We conclude, f cannot have 2 or more roots, so it has exactly one.

12. $n_1 + 2n_2 = 500$

want $n_1 n_2$ maximal

$n_1 = 500 \Rightarrow n_2 = 0, \quad n_1 n_2 = 0$ } endpts.

$n_2 = 250 \Rightarrow n_1 = 0, \quad n_1 n_2 = 0$

$$P(n_1) = n_1 n_2(n_1) = n_1\left(250 - \frac{n_1}{2}\right)$$

$$= 250 n_1 - \frac{n_1^2}{2}$$

$$P'(n_1) = 250 - n_1 = 0$$

$\Rightarrow n_1 = 250, \quad n_2 = 125.$