# Practice Exam #3
# Math 1450

**Fall 2014**
**Section** ___________

**Name:** ______________________________ **Student ID:** ______________________

1. State the precise meaning of the statement the sequence $\{a_n\}_{n=1}^{\infty}$ has the limit

$$\lim_{n\to\infty} a_n = L\,.$$

2. Find the area inside the circle

$$x^2 + y^2 = r^2$$

by using integration. *Hint: It may simplify the problem to calculate the area in the quadrant* $x, y \geq 0$ *and multiply by* $4$.

3. Find the area bounded by the curves $y = \cos^2(x)$, $y = \sin^2(x)$, $x = 0$ and $x = \frac{\pi}{4}$.

4. A solid is obtained by rotating the region bounded by $y = 2x^{1/3}$ and $y = 2x^3$ about the $y$-axis.

   Set-up the integrals that would calculate the volume of the solid by (a) the method of cylindrical shells and (b) the washer method. **You do not need to evaluate the integrals, merely correctly formulate them.**

5. The area bounded by $y = 4 - x^2$, $x = 2$, $y = 4$ is rotated about the $y$ axis. Find the volume of the resulting solid by both (a) method of cylindrical shells and (b) the washer method. **In this case you do need to evaluate the integral you set-up.**

6. Find the arc-length of $y = x^2 - \frac{1}{8}\ln x$ from $x = 1$ to $x = e$.

7. Find the area of the surface generated by rotating the curve $y = \ln x$, between $x = 1$ and $x = e$ about the $y$-axis.

8. Determine whether the following sequences are convergent and if possible, compute their limit.

(i) $a_n = (1 + \frac{2}{n})^n$

(ii) $a_n = \sin(2n\pi/(1 + 8n))$

(iii) $a_n = \ln(2n + 1) - \ln(n + 3)$

9. Explain the truth or falsity of the following two statements, giving examples or proofs (if applicable):

(i) If $\lim_{n\to\infty} a_n = 0$ then $\sum_{n=0}^{\infty} a_n$ converges.

(ii) If $\sum_{n=0}^{\infty} a_n$ converges and all $a_n \geq 0$, then $\lim_{n\to\infty} a_n = 0$.

10. Determine whether the following series converge. State precisely your reasons.

(a)

$$\sum_{n=1}^{\infty} \frac{n}{e^n}$$

(b)

$$\sum_{n=2}^{\infty} \frac{1}{(\ln(n))^2}$$

*Hint:* $\lim_{n\to\infty} \frac{\ln(n)}{n^p} = 0$ *for any* $p > 0$.

*(c)*

$$\frac{1}{1.1} + \frac{1}{2.22} + \frac{1}{3.333} + \frac{1}{4.4444} + \dots$$

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1. State the precise meaning of the statement the sequence $\{a_n\}_{n=1}^{\infty}$ has the limit

$$\lim_{n\to\infty} a_n = L\,.$$

For any given $\varepsilon > 0$ there is $N$
such that for all $n > N$,

$$|a_n - L| < \varepsilon.$$

2. Find the area inside the circle

$$x^2 + y^2 = r^2$$

by using integration. *Hint: It may simplify the problem to calculate the area in the quadrant* $x, y \geq 0$ *and multiply by* $4$.

Area

$$A = 2 \int_{-r}^{r} \sqrt{r^2 - x^2} \, dx$$

$$= 4 \int_{0}^{r} \sqrt{r^2 - x^2} \, dx \qquad x = r \sin\theta \qquad dx = r \cos\theta \, d\theta$$

$$= 4r \int_{0}^{\pi/2} \underbrace{\sqrt{r^2 - r^2 \sin^2\theta}}_{r \cos\theta} \cos\theta \, d\theta$$

$$= 4r^2 \int_{0}^{\pi/2} \underbrace{\cos^2\theta}_{\frac{1}{2} + \frac{1}{2}\cos(2\theta)} \, d\theta$$

$$= 4r^2 \left[ \frac{1}{2}\theta + \frac{1}{4}\sin(2\theta) \right]_0^{\pi/2}$$

$$= \cancel{4} r^2 \frac{\pi}{\cancel{4}} = \pi r^2 .$$

3. Find the area bounded by the curves $y = \cos^2(x)$, $y = \sin^2(x)$, $x = 0$ and $x = \frac{\pi}{4}$.

$$y = \cos^2 x = \frac{1}{2} + \frac{1}{2}\cos(2x)$$

$$y = \sin^2 x = \frac{1}{2} - \frac{1}{2}\cos(2x)$$

Intersection at $\cos(2x) = 0 \Rightarrow 2x = \frac{\pi}{2}$

$$\Rightarrow x = \frac{\pi}{4}$$

For $x$ in $[0, \frac{\pi}{4}]$, $\cancel{\cos^2 x}$ $\cos(2x) \geq 0$,

so $\cos^2(x) \geq \sin^2(x)$.

Area

$$A = \int_0^{\pi/4} (\cos^2 x - \sin^2 x)\,dx$$

$$= \int_0^{\pi/4} \left(\cancel{\tfrac{1}{2}} + \frac{1}{2}\cos(2x) - \cancel{\tfrac{1}{2}} + \frac{1}{2}\cos(2x)\right) dx$$

$$= \int_0^{\pi/4} \cos(2x)\,dx = \frac{1}{2}\sin(2x)\Big|_0^{\frac{\pi}{4}}$$

$$= \frac{1}{2}\sin\left(\frac{\pi}{2}\right) - 0 = \frac{1}{2}.$$

4. A solid is obtained by rotating the region bounded by $y = 2x^{1/3}$ and $y = 2x^3$ about the $y$-axis.

Set-up the integrals that would calculate the volume of the solid by (a) the method of cylindrical shells and (b) the washer method. **You do not need to evaluate the integrals, merely correctly formulate them.**

Intersection point(s) :

$$y = 2x^{\frac{1}{3}} = 2x^3$$

$$x^{\frac{1}{3}} = x^3 \iff x = x^9, \quad \underbrace{x = \pm 1, 0}_{2 \text{ pieces!}}$$

For $|x| \leq 1$, $|x|^{1/3} \geq |x|^3$, so

a) $$V = 2\pi \int_{-1}^{1} \underbrace{2\left|x^{\frac{1}{3}} - x^3\right|}_{\text{height}} \underbrace{|x|}_{\text{radius}} \, dx$$

$$= 8\pi \int_0^1 \left(x^{\frac{1}{3}} - x^3\right) x \, dx$$

b) Solve for $x$ : $$x = \frac{1}{2^3} y^3 = \left(\frac{y}{2}\right)^3$$

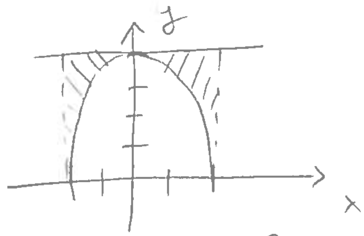
$$x = \frac{1}{2^{1/3}} y^{\frac{1}{3}} = \left(\frac{y}{2}\right)^{\frac{1}{3}}$$

If $x = \pm 1$, $y = \pm 2$

$$V = \pi \int_{-2}^{2} \left( \left|\frac{y}{2}\right|^{\frac{2}{3}} - \left|\frac{y}{2}\right|^6 \right) dy$$

$$= 2\pi \int_0^2 \left( \left(\frac{y}{2}\right)^{\frac{2}{3}} - \left(\frac{y}{2}\right)^6 \right) dy$$

5. The area bounded by $y = 4 - x^2$, $x = 2$, $y = 4$ is rotated about the $y$ axis. Find the volume of the resulting solid by both (a) method of cylindrical shells and (b) the washer method. **In this case you do need to evaluate the integral you set-up**.



a) $V = 2\pi \int_0^2 x\,(4 - (4 - x^2))\,dx$

$= 2\pi \int_0^2 x^3\,dx = 2\pi \left.\frac{x^4}{4}\right|_0^2 = 8\pi$

b)

$V = \pi \int_0^4 \left((2)^2 - (\sqrt{4-y})^2\right) dy$

$= \pi \int_0^4 y\,dy = \pi \left.\frac{y^2}{2}\right|_0^4 = 8\pi$

6. Find the arc-length of $y = x^2 - \frac{1}{8}\ln x$ from $x = 1$ to $x = e$.

$$y' = 2x - \frac{1}{8x}$$

$$L = \int_1^e \sqrt{\underbrace{1 + \left(2x - \frac{1}{8x}\right)^2}_{1 + 4x^2 - 4\frac{1}{8} + \frac{1}{64x^2}}}\, dx$$

$$= \int_1^e \sqrt{\underbrace{4x^2 + \frac{1}{2} + \frac{}{64x^2}}_{\left(2x + \frac{1}{8x}\right)^2}}\, dx$$

$$= \int_1^e \left(2x + \frac{1}{8x}\right) dx$$

$$= \left[x^2 + \frac{1}{8}\ell x\right]_1^e$$

$$= e^2 + \frac{1}{8} - 1 \qquad = e^2 - \frac{7}{8}.$$

7. Find the area of the surface generated by rotating the curve $y = \ln x$, between $x = 1$ and $x = e$ about the $y$-axis.

$$x = 1 \Rightarrow y = 0, \qquad x = e \Rightarrow y = 1$$

$$S = 2\pi \int_0^1 x(y) \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

$$= 2\pi \int_0^1 e^y \sqrt{1 + e^{2y}}\, dy \qquad u = e^y, \quad du = e^y\, dy$$

$$= 2\pi \int_1^e \sqrt{1+u^2}\, du \qquad u = \tan\theta, \quad du = \sec^2\theta\, d\theta$$

$$= 2\pi \int_{\pi/4}^{\tan^{-1}(e)} \sec^3\theta\, d\theta$$

$$= 2\pi \, \frac{1}{2} \Big[ \sec\theta \tan\theta + \ln|\sec\theta + \tan\theta| \Big]_{\pi/4}^{\tan^{-1}(e)}$$

$$= \pi \Big[ \sqrt{1+e^2}\, e + \ln\left|\sqrt{1+e^2} + e\right| - \sqrt{2}(1) - \ln\left|\sqrt{2} + 1\right| \Big]$$

8. Determine whether the following sequences are convergent and if possible, compute their limit.

(i) $a_n = (1 + \frac{2}{n})^n$

(ii) $a_n = \sin(2n\pi/(1+8n))$

(iii) $a_n = \ln(2n+1) - \ln(n+3)$

(i) $a_n = e^{n \ln(1+\frac{2}{n})}$

$$\lim_{n\to\infty} e^{n \ln(1+\frac{2}{n})} = e^{\boxed{\lim_{n\to\infty} n \ln(1+\frac{2}{n})}}$$

$$\lim_{n\to\infty} \frac{\ln(1+\frac{2}{n})}{\frac{1}{n}} \overset{l'H}{=} \lim_{n\to\infty} \frac{\frac{1}{1+\frac{2}{n}}\left(-\frac{2}{\cancel{n^2}}\right)}{-\frac{1}{\cancel{n^2}}} = 2$$

$$\Rightarrow \lim_{n\to\infty} a_n = e^2$$

(ii) $a_n = \sin\left(\frac{2n\pi}{1+8n}\right)$

$$= \sin\left(\frac{2\pi}{8+\frac{1}{n}}\right)$$

$$\lim_{n\to\infty} \sin\left(\frac{2\pi}{8+\frac{1}{n}}\right) = \sin\left(\lim_{n\to\infty} \frac{2\pi}{8+\frac{1}{n}}\right)$$

$$= \sin\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}.$$

(iii) $a_n = \ln\left(\frac{2n+1}{n+3}\right)$

$$= \ln\left(2\left(1 - \underbrace{\frac{5/2}{n+3}}_{\to 0}\right)\right)$$

$$\lim_{n \to \infty} a_n = \ln 2$$

9. Explain the truth or falsity of the following two statements, giving examples or proofs (if applicable):

(i) If $\lim_{n\to\infty} a_n = 0$ then $\sum_{n=0}^{\infty} a_n$ converges.

(ii) If $\sum_{n=0}^{\infty} a_n$ converges and all $a_n \geq 0$, then $\lim_{n\to\infty} a_n = 0$.

(i) No, counterexample $a_n = \frac{1}{n+1}$

then $\sum_{n=0}^{\infty} \frac{1}{n+1} = \sum_{n=1}^{\infty} \frac{1}{n}$ diverges

because by integral comparison with

$f(x) = \frac{1}{x}$ (positive, cont., decreasing)

$\int_1^{\infty} \frac{1}{x}\,dx = \lim_{b\to\infty} [\ln x]_1^b = \infty$

so series diverges.

(ii) Yes, because otherwise for each $\varepsilon = \frac{1}{n}$

exists $m_n$, $m_n > m_{n-1}$, s.th. $a_{m_n} > \frac{1}{n}$,

so $\sum_{n=0}^{\infty} a_n \geq \sum_{n=0}^{\infty} a_{m_n} = \sum_{m=1}^{\infty} \frac{1}{m} = \infty$.

10. Determine whether the following series converge. State precisely your reasons.

(a)

$$\sum_{n=1}^{\infty} \frac{n}{e^n}$$

(b)

$$\sum_{n=2}^{\infty} \frac{1}{(\ln(n))^2}$$

*Hint:* $\lim_{n\to\infty} \frac{\ln(n)}{n^p} = 0$ *for any* $p > 0$.

(c)

$$\frac{1}{1.1} + \frac{1}{2.22} + \frac{1}{3.333} + \frac{1}{4.4444} + \dots$$

a) Integral comparison : $f(x) = x e^{-x}$

$f'(x) = e^{-x} - x e^{-x} = (1-x) e^{-x} \leq 0$

if $x > 1$, $f$ decreasing, pos.

so by

$$\int_1^\infty x e^{-x}\, dx = \lim_{b\to\infty} \int_1^b x e^{-x}\, dx$$

$$= \lim_{b\to\infty} \left[ -x e^{-x} \Big|_1^b + \int_1^b e^{-x}\, dx \right]$$

$$= \lim_{b\to\infty} \left[ \underbrace{-be^{-b}}_{\to 0} + e^{-1} - \underbrace{e^{-b}}_{\to 0} + e^{-1} \right] = \frac{2}{e} < \infty$$

Series converges

b) $\sum_{n=2}^{\infty} \frac{1}{(\ln n)^2}$, compare with

$$\int_2^{\infty} \frac{1}{(\ln x)^2}\,dx = ?$$

By $\frac{1}{x} \le 1$, $\underbrace{\int_1^x \frac{1}{x'}\,dx'}_{=\ln x} \le \int_1^x dx' = x - 1 \le x$

so $\frac{1}{\ln x} \ge \frac{1}{x}$,

$$\int_2^{\infty} \frac{1}{(\ln x)^2}\,dx \ge \int_2^{\infty} \frac{1}{x \ln x}\,dx = \int_{\ln 2}^{\infty} \frac{1}{u}\,du$$

$$= \lim_{b\to\infty} \left[\ln b - \ln \ln 2\right]$$

$$= \infty.$$

Thus, series diverges.

c) $a_1 = \frac{1}{1.1} = \left(1 + \frac{1}{10}\right)^{-1}$

$a_2 = \frac{1}{2.22} = \left(2\left(1 + \frac{1}{10} + \frac{1}{100}\right)\right)^{-1}$

...

$$a_n = \left(n\left(1 + \frac{1}{10} + \dots + \frac{1}{10^{n}}\right)\right)^{-1}$$

$$= \left(n \frac{1 - \frac{1}{10^{n+1}}}{1 - \frac{1}{10}}\right)^{-1}$$

$$= \frac{1 - \frac{1}{10}}{n\left(1 - \frac{1}{10^{n+1}}\right)}$$

compare with

$$f(x) = \frac{1 - \frac{1}{10}}{x\left(1 - \frac{1}{10^{x+1}}\right)} \quad \left\{\begin{array}{l} x \text{ increasing} \\ 1 - \frac{1}{10^{x+1}} \text{ increasing} \\ \Rightarrow f \text{ decreasing} \end{array}\right.$$

By $1 - \frac{1}{10^{x+1}} < 1$,

$$\frac{1 - \frac{1}{10}}{x\left(1 - \frac{1}{10^{x+1}}\right)} > \frac{1 - \frac{1}{10}}{x}$$

$$\Rightarrow \int_1^\infty f(x)\,dx \geq \int_1^\infty \frac{9/10}{x}\,dx$$

$$= \infty.$$

So series diverges.