Math1450 Accelerated Calculus, Practice Final Exam

Please write your answers clearly and in a logical and well-organized way. Points will be deducted for sloppy work. Attempt all questions. You have 160 minutes.

Good Luck!

(1) [10 points] Find the absolute maximum, absolute minimum, all local extremal values and all critical points of

$$f(x) = \frac{x+1}{x^2+1}$$

on the closed interval $[-1, \frac{1}{2}]$.

(2)[10 points] Show, using the mean value theorem or otherwise, that if $f(x)$ is twice differentiable on $\mathbb{R}$ and $f(x)$ has 3 distinct roots then $f'(x)$ has 2 distinct roots and $f''(x)$ has at least one root.

(3) [10 points] A closed cylindrical can is to have a volume of V cubic centimeters. Find the radius and height of such a can which gives the minimum surface area. Note that the surface area of the top and bottom are to be included. *Make sure to show all working. Your answer will involve the parameter* V.

(4) [10 points] (a) The pressure P and volume V of a gas are related under adiabatic expansion by

$$PV^{3/2} = k$$

where k is a constant. Suppose that at some instant the volume of the gas is 4 units, the pressure is 5 units and the pressure is increasing at 0.5 units per second. Find the rate of change of the volume of the gas at that instant.

(b) State without proof the following limits:

(i) $\lim_{x\to 0} x\ln(x^2)$

(ii) $\lim_{x\to\infty} \frac{2x+1}{x+3\sqrt{x}+1}$

(iii) $\lim_{x\to 0} e^x \cos(x)$

(iv) $\lim_{x\to\infty} e^{-\sqrt{3}x}(x^{10}+12x^2)$

(5) [10 points] The rate of decay of uranium with respect to time is proportional to the mass of uranium.

(a) Write down an equation expressing the statement above.

(b) If it takes one years for a piece of uranium decrease in mass from 10 grams to 6 grams, what mass will be left after another 2 years?

(6)[20 points] Evaluate the following indefinite integrals:

(a) $\int \frac{\sqrt{2}}{x^2+1}\, dx$

(b) $\int \frac{(x-1)^2}{(x-2)^2}\, dx$

(c) $\int \ln x \, dx$

(d) $\int x^2 e^x \, dx$

(7) [10 points] The area bounded by $y = x^{2/3}$, $x = 5$, $y = 1$ is rotated about the y axis. Find the volume of the resulting solid by both (a) method of cylindrical shells and (b) the method of cross-sectional area. **You do need to evaluate the integral you set-up.**

(8)[10 points] Find the area bounded by the curves $x = y^2$ and $y = x$.

(9)[10 points] (a) By comparing to a simpler integrand show that

$$\int_0^1 e^{-x^2}\, dx \geq 1 - e$$

(b) Suppose that $f(0) = 1$ and $f'(x) \leq x$ for all $x \geq 0$. Show that $f(1) \leq \frac{3}{2}$.

(c) Suppose for $x \geq 0$,

$$F(x) = \int_0^{\sqrt{x}} t^2\, dt$$

Find $F'(4)$.

(10) [10 points] (a) Find

$$\int_0^{\pi/2} \frac{\sin(\sqrt{x})}{\sqrt{x}}\, dx$$

(b) Does

$$\lim_{x\to 0} \int_x^{\pi/2} \frac{1}{\sin(t)}\, dt$$

exist? Briefly justify your answer. Note that you do not need to evaluate the integral.

(c) Does

$$\lim_{x\to 0} \int_x^{\pi/2} \frac{t}{\sin(t)}\, dt$$

exist? Briefly justify your answer. Note that you do not need to evaluate the integral.

(11) [20 points] Determine whether or not the following series converge, giving your reasons in detail.

(a)

$$\sum_{n=1}^{\infty} \frac{4^n}{(n)!}$$

(b)

$$\sum_{n=1}^{\infty} \frac{n^3}{3^n}$$

(c)

$$\sum_{n=2}^{\infty} \frac{2}{\sqrt{n}}$$

(d)

$$\sum_{n=2}^{\infty} \frac{(-1)^{n+3}}{2n + \sqrt{n}}$$

(12) Determine whether or not the following infinite series converge. If they do converge find their sum.

(a) [5 points]

$$\sum_{n=3}^{\infty} \frac{2}{3^n}$$

(b) [5 points]

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$$

Hint for (b): Using partial fractions write $\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$

(13) [10 points] (i) Find the power series expansion of the function

$$\frac{1}{1-x}$$

about $a = 0$.

(ii) Using (i) or otherwise find the Taylor series expansion of

$$\frac{1}{(1-x)^2}$$

and

$$\frac{1}{(1-x)^3}$$

about $a = 0$, stating carefully any theorems you may use about integrating or differentiating power series within their radius of convergence.

(14) [10 points] Find the 3rd order Taylor polynomial of the following function about the indicated point.

$$f(x) = \frac{1}{\sqrt{x}}, \quad a = 9$$

(15) (a) [5 points] Describe the behavior of a function $f(x)$ defined by a power series

$$f(x) = \sum_{n=1}^{\infty} a_n (x-a)^n$$

with regard to differentiability and integrability at points $|x-a| < R$ where R is the radius of convergence.

(b)[6 points] Using (a) or otherwise find the Taylor expansion of

$$\tan^{-1}(x)$$

about $a = 0$. *Hint:* $\frac{d}{dx}\tan^{-1}(x) = \frac{1}{1+x^2}$.

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(1) $f(x) = \dfrac{x+1}{x^2+1}$

$$f'(x) = \frac{1}{x^2+1} - \frac{x+1}{(x^2+1)^2}(2x) = \frac{x^2+1-2x^2-2x}{(x^2+1)^2}$$

$$= - \frac{(x+1+\sqrt{2})(x+1-\sqrt{2})}{(x^2+1)^2}$$

x	$-1-\sqrt{2}$	-1	$-1+\sqrt{2}$	$1/2$
sign f'	$-$		$+$	$-$

f increasing on $(-1, -1+\sqrt{2})$, decreasing on $(-1+\sqrt{2}, \frac{1}{2})$, crit pt. $x = -1+\sqrt{2}$

Endpoint values $f(-1) = \frac{0}{2} = 0$,

$f(\frac{1}{2}) = \dfrac{\frac{1}{2}+1}{\frac{1}{4}+1} = \frac{6}{5}$

value at crit pt. $f(-1+\sqrt{2}) = \dfrac{\sqrt{2}}{4-2\sqrt{2}}$

global max is $\dfrac{\sqrt{2}}{4-2\sqrt{2}}$, also local max,

global min is 0 ($< \frac{6}{5}$).

(2) If f is twice diff'able and f has 3 distinct roots x_1, x_2, x_3

assuming $x_1 < x_2 < x_3$, then by Rolle's theorem, there is y_1 in (x_1, x_2) and y_2 in (x_2, x_3) with $f'(y_1) = f'(y_2) = 0$

Since f' is differentiable, applying Rolle again gives z_1 in (y_1, y_2) with

$(f')'(z_1) = f''(z_1) = 0$.

(3) Volume $V = \pi r^2 h \Rightarrow h = \frac{V}{\pi r^2}$

Surface $A = 2\pi r^2 + 2\pi r h$

$$A(r) = 2\pi r^2 + 2\pi r \frac{V}{\pi r^2}$$

$$A'(r) = 4\pi r - 2\frac{V}{r^2} = 0$$

$$r^3 = \frac{V}{2\pi}$$

Only one critical value for $r > 0$

and $\lim_{r \to 0^+} A(r) = \infty$, $\lim_{r \to \infty} A(r) = \infty$

so global min achieved at crit. pt.

$$r = \frac{V^{\frac{1}{3}}}{(2\pi)^{\frac{1}{3}}}, \quad h = \frac{V}{\pi r^2} = \frac{2^{\frac{2}{3}} V^{\frac{1}{3}}}{\pi^{1/3}}.$$

(4) a) $PV^{3/2} = k$, $\quad \frac{dP}{dt} = 0.5 \frac{1}{s}$

$$V^{3/2} = \frac{k}{P}$$

$$\frac{d}{dt} V^{3/2} = \frac{3}{2} V^{\frac{1}{2}} \frac{dV}{dt} = -\frac{k}{P^2} \frac{dP}{dt}$$

$$\Rightarrow \frac{dV}{dt} = -\frac{2}{3} \frac{1}{V^{\frac{1}{2}}} \frac{k}{P^2} \frac{dP}{dt}$$

$$= -\frac{2}{3} \frac{1}{\sqrt{4}} \frac{k}{25} (0.5) \frac{1}{s}$$

$$= -\frac{k}{150} \frac{1}{s}.$$

b) i) $\lim_{x\to 0^+} x \ln x^2 = \lim_{x\to 0^+} 2x \ln x = 0$

ii) $\lim_{x\to\infty} \frac{2x+1}{x+3\sqrt{x}+1} = 2$

iii) $\lim_{x\to 0} e^x \cos(x) = 1$

iv) $\lim_{x\to\infty} e^{-\sqrt{3}x}(x^{10}+12x^2) = 0$

(5) a) $\frac{dM}{dt} = -kM$

$\Rightarrow M(t) = M(0)e^{-kt}$, $M(0) = 10$

$M(1) = M(0)\underbrace{e^{-k(1)}}_{\frac{6}{10}} = 6$

$M(3) = M(0)e^{-k(3)}$

$= \underbrace{M(0)}_{10}\underbrace{(e^{-k})^3}_{\left(\frac{6}{10}\right)^3}$

$= \frac{6(36)}{100} = \frac{54}{25}$

(6) (a) $\int \frac{\sqrt{2}}{x^2+1}\, dx = \sqrt{2} \arctan(x) + C$

(b) $\int \frac{(x-1)^2}{(x-2)^2}\, dx$ $\qquad u = x-2$

$= \int \frac{(u+1)^2}{u^2}\, du = \int \left(1 + \frac{2}{u} + \frac{1}{u^2}\right) du$

$= u + 2\ln|u| - u^{-1} + C$

$= x + 2\ln|x-2| - \frac{1}{x-2} + C$

c) $\int \ln x\, dx = x\ln x - \int x \frac{1}{x}\, dx$

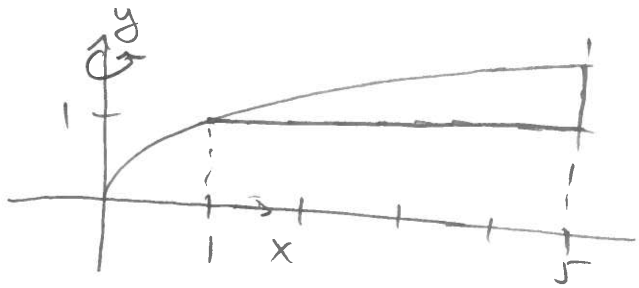
$= x\ln x - x + C$

d) $\int x^2 e^x\, dx = x^2 e^x - \int (2x) e^x\, dx$

$= x^2 e^x - (2x)e^x + \int (2) e^x\, dx$

$= (x^2 - 2x + 2) e^x + C$

(7)



a) $$V = 2\pi \int_1^5 x\left(x^{2/3} - 1\right) dx$$

$$= 2\pi \left[\frac{3}{8} x^{8/3} - \frac{1}{2} x^2 \right]_1^5$$

$$= 2\pi \left(\frac{3}{8} 5^{8/3} - \frac{1}{2}(25) - \frac{3}{8} + \frac{1}{2} \right)$$

$$= 2\pi \left(\frac{3}{8}\left(5^{8/3} - 1\right) - 12 \right) = \pi \left(\frac{3}{4} 5^{8/3} - 24\frac{3}{4} \right)$$

b) Limits $y = 1$, $y = 5^{2/3}$, $x = y^{3/2}$

$$V = \pi \int_1^{5^{2/3}} \left(5^2 - y^3 \right) dy$$

$$= \pi \left(25\left(5^{2/3} - 1\right) - \frac{1}{4} y^4 \Big|_1^{5^{2/3}} \right)$$

$$= \pi \left(\cancel{25}\, \cancel{5^{2/3}}\ 5^{8/3} - 25 - \frac{1}{4} 5^{8/3} + \frac{1}{4} \right)$$

$$= \pi \left(\frac{3}{4} 5^{8/3} - 24\frac{3}{4} \right)$$

(8) $A = \int_0^1 (y - y^2)\,dy$

$= \left[\frac{1}{2} y^2 - \frac{1}{3} y^3 \right]_0^1$

$= \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$.

(9) (a) $\int_0^1 e^{-x^2}\,dx \geq \int_0^1 e^{-x}\,dx = 1 - e$

because of $0 \leq x \leq 1$,

$x^2 \leq x$

$-x^2 \geq -x$

$e^{-x^2} \geq e^{-x}$.

(b) $f(0) = 1$, $f'(x) \leq x$

$f(1) - f(0) = \int_0^1 f'(x)\,dx$

$\leq \int_0^1 x\,dx = \frac{1}{2}(1)^2 - \frac{1}{2}(0)^2$

$f(1) \leq f(0) + \frac{1}{2} = \frac{3}{2}$

(c) $$F(x) = \int_0^{\sqrt{x}} t^2\,dt$$

$$\frac{dF}{dx} = \frac{d}{du}\int_0^{u} t^2\,dt \;\frac{du}{dx}, \quad u = \sqrt{x}$$

$$\overset{FTC}{=} u^2 \;\frac{1}{2}\;\frac{1}{\sqrt{x}} = \frac{x}{2\sqrt{x}} = \frac{\sqrt{\cancel{x}}}{\cancel{x}2}.$$

$$F'(4) = \frac{\sqrt{4}}{2} = 1.$$

(10) a) $$\int_0^{\pi/2} \frac{\sin(\sqrt{x})}{\sqrt{x}}\,dx = 2\int_0^{\sqrt{\pi/2}} \sin(u)\,du \qquad u=\sqrt{x}$$

$$= -2\cos\left(\sqrt{\pi/2}\right) + 2\underbrace{\cos(0)}_{1}$$

$$= 2 - 2\cos\left(\sqrt{\pi/2}\right)$$

b) We know $\sin(t) \le t$, so $\frac{1}{t} \le \frac{1}{\sin t}$

and $\lim_{x\to 0^+} \int_x^{\pi/2} \frac{1}{t} = \infty$, so by int. comparison

$$\lim_{x\to 0^+} \int_x^{\pi/2} \frac{1}{\sin t}\,dt = \infty \quad \text{as well.}$$

c) $$\lim_{x\to 0} \int_x^{\pi/2} \frac{t}{\sin t}\, dt = ?$$

integrand is continuous and

$$\lim_{x\to 0} \frac{x}{\sin x} = 1$$

exists, so integral exists/converges as usual Riemann integral.

(11) a) $$\sum_{n=1}^{\infty} \frac{4^n}{n!} \qquad a_n = \frac{4^n}{n!}$$

quot. test $$\left|\frac{a_{n+1}}{a_n}\right| = \frac{4^{n+1}}{(n+1)!} \frac{n!}{4^n}$$

$$= \frac{4}{n+1} \xrightarrow{n\to 0} 0$$

converges

b) $$\sum_{n=1}^{\infty} \frac{n^3}{3^n} \qquad a_n = \frac{n^3}{3^n}$$

quot. test $$\left|\frac{a_{n+1}}{a_n}\right| = \frac{(n+1)^3}{3^{n+1}} \frac{3^n}{n^3}$$

$$\xrightarrow{n\to\infty} \frac{1}{3} < 1$$

converges

(c) $\sum_{n=2}^{\infty} \frac{2}{\sqrt{n}}$ diverges, p-series with $p = \frac{1}{2}$

(d) $\sum_{n=2}^{\infty} (-1)^{n+3} \underbrace{\frac{1}{2n+\sqrt{n}}}$

decreasing to 0 b/c $2n+\sqrt{n}$ inc. to ∞

converges by alt. series test

(12) a) $\sum_{n=3}^{\infty} \frac{2}{3^n} = 2 \sum_{n=3}^{\infty} \frac{1}{3^n} = 2 \left(\sum_{n=0}^{\infty} \frac{1}{3^n} - 1 - \frac{1}{3} - \frac{1}{3^2} \right)$

$$= 2 \left(\underbrace{\frac{1}{1-\frac{1}{3}}}_{\frac{3}{2}} - \frac{4}{3} - \frac{1}{9} \right)$$

$$= 2 \left(\frac{27}{18} - \frac{24}{18} - \frac{2}{18} \right) = \frac{2}{18} = \frac{1}{9}.$$

b) $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{1}{i} - \frac{1}{i+1} \right)$

$$= \frac{1}{1} - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \dots + \frac{1}{n} - \frac{1}{n+1}$$

$$= \lim_{n\to\infty} \left(1 - \frac{1}{n+1} \right) = 1.$$

(13) (i) $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ (geom. series)

(ii) $f(x) = \frac{1}{1-x}$, $f'(x) = +\frac{1}{(1-x)^2}$

$f''(x) = (+2)\frac{1}{(1-x)^3}$

We can differentiate term by term within radius of conv.

Here, radius is $r = 1$.

So,

$$\frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} n x^{n-1}$$

and

$$\frac{1}{(1-x)^3} = \frac{1}{2} \sum_{n=2}^{\infty} n(n-1) x^{n-2}$$

(14) $f(x) = \frac{1}{\sqrt{x}}$, $f'(x) = -\frac{1}{2} x^{-\frac{3}{2}}$

$f''(x) = (-\frac{1}{2})(-\frac{3}{2}) x^{-\frac{5}{2}}$

$f'''(x) = (-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2}) x^{-\frac{7}{2}}$

Taylor poly at $a = 9$,

$$T_3 f(x) = \frac{1}{\sqrt{9}} + (x-9)(-\frac{1}{2})(9^{-\frac{3}{2}})$$

$$+ \frac{(x-9)^2}{2}(-\frac{1}{2})(-\frac{3}{2})(9^{-\frac{5}{2}})$$

$$+ \frac{(x-9)^3}{2\,\cancel{3!}}(\cancel{+}\frac{1}{2})(\cancel{+}\frac{\cancel{3}}{2})(-\frac{5}{2})(9^{-\frac{7}{2}})$$

$$= \frac{1}{3} + (x-9)(-\frac{1}{54})$$

$$+ (x-9)^2 \frac{1}{648}$$

$$\cancel{+} - (x-9)^3 \frac{5}{16} \frac{1}{3^7}$$

(15) a) The power series of the derivative or integral has the same radius of convergence. Differentiation or integration can be performed term by term, for all x, $|x-a| < R$, so

$$\int f(x)\,dx = \sum_{n=1}^{\infty} \frac{a_n}{n+1} (x-a)^{n+1},$$

$$f'(x) = \sum_{n=1}^{\infty} n\, a_n (x-a)^{n-1}.$$

b) $$\frac{d}{dx} \tan^{-1}(x) = \frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n (x^2)^n$$

so integrating term by term gives

$$\tan^{-1}(x) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{2n+1} x^{2n+1}.$$