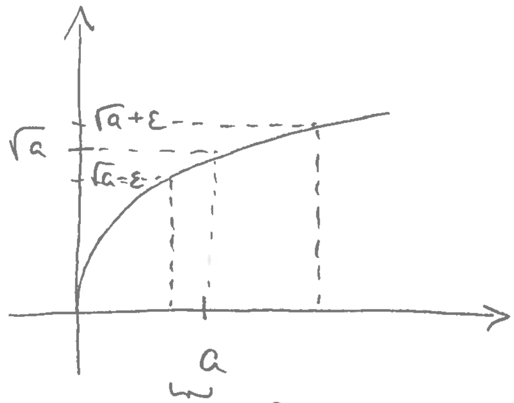
2.4 #37



$$\underbrace{\quad}_{\delta = ?}$$

Book suggestion : $\quad |\sqrt{x} - \sqrt{a}| = \dfrac{|x-a|}{\sqrt{x}+\sqrt{a}} \overset{?}{<} \varepsilon$

Problem : $\quad \underbrace{|x-a|}_{\text{good}} < \varepsilon(\sqrt{x} + \sqrt{a})$

do <u>not</u> want x on RHS !

How to get rid of $\sqrt{x}$? Can always make δ <u>smaller</u>, so assume $\delta \le \frac{a}{2}$, then

$|x-a| < \delta \implies \frac{a}{2} = a - \frac{a}{2} < x \left(< \frac{3a}{2}\right)$, so $\sqrt{\frac{a}{2}} < \sqrt{x}$.

Given $\varepsilon > 0$, choose $\delta = \min\left\{ \frac{a}{2}, \varepsilon\left(\sqrt{\frac{a}{2}} + \sqrt{a}\right)\right\}$

then gives that for $0 < |x-a| < \delta$, we have

$$|\sqrt{x} - \sqrt{a}| = \frac{|x-a|}{\sqrt{x}+\sqrt{a}} < \frac{\delta}{\sqrt{x}+\sqrt{a}}$$

$$\le \frac{\varepsilon\left(\sqrt{\frac{a}{2}} + \sqrt{a}\right)}{\sqrt{x}+\sqrt{a}} \le \varepsilon \cancel{\frac{\sqrt{x}+\sqrt{a}}{\sqrt{x}+\sqrt{a}}}.$$

Alternative solution: (See sketch above)

To get $\sqrt{a} - \varepsilon < \sqrt{x} < \sqrt{a} + \varepsilon$

we need $(\sqrt{a} - \varepsilon)^2 < x < (\sqrt{a} + \varepsilon)$

$$a - 2\sqrt{a}\varepsilon + \varepsilon^2 < x < a + 2\sqrt{a}\varepsilon + \varepsilon^2 .$$

Want $-(2\sqrt{a}\varepsilon - \varepsilon^2) < x - a < 2\sqrt{a}\varepsilon + \varepsilon^2$

Choose $\delta = \min\{2\sqrt{a}\,\varepsilon - \varepsilon^2,\ 2\sqrt{a}\,\varepsilon + \varepsilon^2\}$

$$= 2\sqrt{a}\,\varepsilon - \varepsilon^2 .$$

Given $\varepsilon > 0$, then choosing δ as indicated then gives that if $0 < |x - a| < \delta$,

$$-(2\sqrt{a}\varepsilon - \varepsilon^2) < x - a < 2\sqrt{a}\varepsilon - \varepsilon^2 < 2\sqrt{a}\varepsilon + \varepsilon^2$$

$$\Longrightarrow \quad a - 2\sqrt{a}\varepsilon + \varepsilon^2 < x < a + 2\sqrt{a}\varepsilon + \varepsilon^2$$

$$\Longrightarrow \quad (\sqrt{a} - \varepsilon)^2 < x < (\sqrt{a} + \varepsilon)^2$$

$$\Longrightarrow \quad \sqrt{a} - \varepsilon < \sqrt{x} < \sqrt{a} + \varepsilon$$

$$\Longrightarrow \quad |\sqrt{x} - \sqrt{a}| < \varepsilon .$$