Practice Exam # 1
Math 1451

Spring 2015
Section __________

Name: ____________________________ Last 4 digits of student ID: ________

The use of calculators or cell phones is not allowed in this exam. Show all the required work to obtain full credit. If running out of space, you may use the back sides of the pages or the empty page at the end. The duration of this exam is 80 minutes.

(1) (a) Find the equation of the line which is normal to the vectors $\vec{u} = \langle 1, -1, 0 \rangle$ and $\vec{v} = \langle 2, 0, 1 \rangle$ and which contains the point $P = (1, 1, 1)$.

(b) Find the parametric equation of the line $\vec{r}(t)$, $0 \leq t \leq 1$, between $P = (2, -3, 1)$ and $Q = (-2, 1, 7)$.

(2) (a) Show that if $\vec{r}$ is a vector-valued function with values in $\mathbb{R}^3$ and $\vec{r}'(t) \cdot \vec{r}'(t) = C$ for all t, where C is a constant, then $\vec{r}''(t) \cdot r'(t) = 0$ for all t.

(b) Show that if $\vec{r}$ is a vector-valued function with values in $\mathbb{R}^3$ and $\vec{r}''(t) = -f(\vec{r}(t))\vec{r}(t)$ where $f : \mathbb{R}^3 \to \mathbb{R}$ is a scalar function then $\vec{r}(t) \times \vec{r}'(t) = C$ for some constant C.

(3) (a) Find the distance of the point $(1, -1, 2)$ to the plane $3x + 2y + 6z = 5$.

(b) Find the distance between the planes $2x - 4y + z = -2$ and $x - 2y + \frac{z}{2} = 2$.

(4) (a) Find the area of the parallelogram determined by the vectors $\vec{u} = \langle 1, 2, 0 \rangle$ and $\vec{v} = \langle 2, 1, 0 \rangle$ in $\mathbb{R}^3$.

(b) What is the component of the vector $\vec{u} = \langle 1, 2, 0 \rangle$ in the direction of the vector $\vec{v} = \langle 2, 1, 0 \rangle$?

(5) Find the first $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$ and second $\frac{\partial^2 f}{\partial x^2}$, $\frac{\partial^2 f}{\partial y^2}$, $\frac{\partial^2 f}{\partial x \partial y}$ partial derivatives of the functions $f(x, y)$:

(a) $f(x, y) = x \sin(xy)$.

(b) $f(x, y) = y\sqrt{1 - x^2}$.

(6) The gas law for an ideal is $PV = nRT$ where P is pressure, V is volume, T is temperature (all in appropriate units) and n is a constant. Show that

$$\frac{\partial P}{\partial V}\frac{\partial V}{\partial T}\frac{\partial T}{\partial P} = -1$$

(7) (a) For the curve with vector equation $\vec{r}(t) = 2\cos(t)\vec{\imath} + 2\sin(t)\vec{\jmath} + t\vec{k}$,

(i) Describe the shape of the curve

(ii) Find the length of the curve from the point $(2, 0, 0)$ to the point $(2, 0, 2\pi)$.

(iii) Find the curvature as a function of t.

(8) (a) Find the equation of the tangent plane to the surface

$$z = 2x^2 - 3y^2 + 2y - x$$

at the point $(-1, 2, -5)$.

(b) Find the linearization $L(x, y)$ of the function

$$f(x, y) = \sin(x + 2y)$$

at the point $(\frac{\pi}{2}, \frac{\pi}{6})$ and use it to estimate $f(\frac{5\pi}{8}, \frac{\pi}{6})$.

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(1) (a) Find the param. vector equation of the line which is normal to the vectors $\vec{u} = \langle 1, -1, 0\rangle$ and $\vec{v} = \langle 2, 0, 1\rangle$ and which contains the point $P = (1, 1, 1)$.

Direction $\vec{n} = \vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -1 & 0 \\ 2 & 0 & 1 \end{vmatrix} = -\vec{i} - \vec{j} + 2\vec{k}$

$= \langle -1, -1, 2\rangle$

Equation $\vec{r} = \langle 1, 1, 1\rangle + t\langle -1, -1, 2\rangle$

(b) Find the parametric vector equation of the line $\vec{r}(t)$, $0 \le t \le 1$, between $P = (2, -3, 1)$ and $Q = (-2, 1, 7)$.

$\vec{r}(t) = (1-t)\langle 2, -3, 1\rangle + t\langle -2, 1, 7\rangle, \quad 0 \le t \le 1$

(2) (a) Show that if $\vec{r}$ is a vector-valued function with values in $\mathbb{R}^3$ and $\vec{r}'(t) \cdot \vec{r}'(t) = C$ for all t, where C is a constant, then $\vec{r}''(t) \cdot r'(t) = 0$ for all t.

By product rule,

$$\frac{d}{dt}\left(\vec{r}'(t) \cdot \vec{r}'(t)\right) = \vec{r}''(t) \cdot \vec{r}'(t) + \vec{r}'(t) \cdot \vec{r}''(t)$$

$$= 2\vec{r}''(t) \cdot \vec{r}'(t) = 0$$

so

$$\vec{r}''(t) \cdot \vec{r}'(t) = 0$$

for all t.

(b) Show that if $\vec{r}$ is a vector-valued function with values in $\mathbb{R}^3$ and $\vec{r}''(t) = -f(\vec{r}(t))\vec{r}(t)$ where $f : \mathbb{R}^3 \to \mathbb{R}$ is a scalar function then $\vec{r}(t) \times \vec{r}'(t) = C$ for some constant C.

We have

$$\frac{d}{dt}\left(\vec{r}(t) \times \vec{r}'(t)\right) = \underbrace{\vec{r}'(t) \times \vec{r}'(t)}_{0} + \vec{r}(t) \times \vec{r}''(t)$$

$$= -\vec{r}(t) \times \left(f(\vec{r}(t))\,\vec{r}(t)\right) = -f(\vec{r}(t))\,\underbrace{\vec{r}(t) \times \vec{r}(t)}_{0}$$

$$= 0$$

so by FTC,

$$\vec{r}(t) \times \vec{r}'(t) = C \quad \text{for some } C.$$

(3) (a) Find the distance of the point $(1, -1, 2)$ to the plane $3x + 2y + 6z = 5$.

We guess P on plane, $P = \langle 1, 1, 0 \rangle$

Distance is projected component onto normal,

$$D = \left| \left(\langle 1, -1, 2 \rangle - \langle 1, 1, 0 \rangle \right) \cdot \frac{\langle 3, 2, 6 \rangle}{\sqrt{9+4+36}} \right|$$

$$= \left| \frac{\langle 0, -2, 2 \rangle \cdot \langle 3, 2, 6 \rangle}{\sqrt{49}} \right|$$

$$= \frac{8}{7}.$$

(b) Find the distance between the planes $2x - 4y + z = -2$ and $x - 2y + \frac{z}{2} = 2$.

Planes have same $\vec{n} = \langle 2, -4, 1 \rangle$,

Unit normal $\vec{u} = \dfrac{\langle 2, -4, 1 \rangle}{\sqrt{21}}$.

Two points, $P = \langle 0, 0, -2 \rangle$

$Q = \langle 0, 0, 4 \rangle$

Difference vector projected onto unit normal

$$D = \left| \left(\langle 0, 0, -2 \rangle - \langle 0, 0, 4 \rangle \right) \cdot \frac{\langle 2, -4, 1 \rangle}{\sqrt{21}} \right|$$

$$= \frac{6}{\sqrt{21}}$$

(4) (a) Find the area of the parallelogram determined by the vectors $\vec{u} = \langle 1, 2, 0\rangle$ and $\vec{v} = \langle 2, 1, 0\rangle$ in $\mathbb{R}^3$.

$$\text{area} = |\vec{u} \times \vec{v}| = \left| \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & 0 \\ 2 & 1 & 0 \end{vmatrix} \right|$$

$$= \left| \cancel{-\vec{j}(0)} \; (1-4)\vec{k} \right| = 3$$

(b) What is the component of the vector $\vec{u} = \langle 1, 2, 0\rangle$ in the direction of the vector $\vec{v} = \langle 2, 1, 0\rangle$?

$$\text{comp}_{\vec{v}} \vec{u} = \vec{u} \cdot \frac{\vec{v}}{|\vec{v}|}$$

$$= \frac{\langle 1, 2, 0\rangle \cdot \langle 2, 1, 0\rangle}{\sqrt{4+1}}$$

$$= \frac{4}{\sqrt{5}}$$

(5) Find the first $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$ and second $\frac{\partial^2 f}{\partial x^2}$, $\frac{\partial^2 f}{\partial y^2}$, $\frac{\partial^2 f}{\partial x \partial y}$ partial derivatives of the functions $f(x, y)$:

(a) $f(x, y) = x \sin(xy)$.

$$f_x = \sin(xy) + xy\cos(xy)$$

$$f_y = x^2 \cos(xy)$$

$$f_{xx} = 2y\cos(xy) + y\,\cancel{\cos(xy)} \mp xy^2 \sin(xy)$$

$$f_{xy} = f_{yx} = 2x\cos(xy) \mp x^2 y \sin(xy)$$

$$f_{yy} = -x^3 \sin(xy)$$

(b) $f(x, y) = y\sqrt{1 - x^2}$.

$$f_x = -\frac{xy}{\sqrt{1-x^2}}$$

$$f_y = \sqrt{1-x^2}$$

$$f_{xx} = -\frac{y}{\sqrt{1-x^2}} \mp \cancel{\tfrac{1}{2}}\,\frac{x^2 y}{\sqrt{1-x^2}^{\,3}} = \frac{-(1-\cancel{x^2})\,y - \cancel{x^2 y}}{\sqrt{1-x^2}^{\,3}}$$

$$f_{xy} = f_{yx} = -\frac{x}{\sqrt{1-x^2}}$$

$$f_{yy} = 0$$

(6) The gas law for an ideal is $PV = nRT$ where P is pressure, V is volume, T is temperature (all in appropriate units) and n is a constant. Show that

$$\frac{\partial P}{\partial V}\frac{\partial V}{\partial T}\frac{\partial T}{\partial P} = -1$$

$$P = \frac{nRT}{V}, \quad V = \frac{nRT}{P}, \quad T = \frac{PV}{nR}$$

$$\frac{\partial P}{\partial V} = -\frac{nRT}{V^2}, \quad \frac{\partial V}{\partial T} = \frac{nR}{P}, \quad \frac{\partial T}{\partial P} = \frac{V}{nR}$$

$$\Rightarrow \quad \frac{\partial P}{\partial V}\frac{\partial V}{\partial T}\frac{\partial T}{\partial P} = -\frac{nRT}{V^{\cancel{2}}}\frac{\cancel{nR}}{P}\frac{\cancel{V}}{\cancel{nR}}$$

$$= -\frac{nRT}{VP} = -1.$$

(7) (a) For the curve with vector equation $\vec{r}(t) = 2\cos(t)\vec{\imath} + 2\sin(t)\vec{\jmath} + t\vec{k}$,

(i) Describe the shape of the curve

Helix inside cylinder $x^2 + y^2 = 4$, radius 2, centered at z-axis.

(ii) Find the length of the curve from the point $(2, 0, 0)$ to the point $(2, 0, 2\pi)$.

$$L = \int_0^{2\pi} |\vec{v}(t)|\, dt, \qquad \vec{v}(t) = \vec{r}\,'(t) = \langle -2\sin t, 2\cos t, 1 \rangle$$

$$|\vec{v}(t)| = \sqrt{4(\sin^2 t + \cos^2 t) + 1} = \sqrt{5}$$

so

$$L = \int_0^{2\pi} \sqrt{5}\, dt = 2\pi\sqrt{5}$$

(iii) Find the curvature as a function of t.

$$\kappa(t) = \frac{|\vec{r}\,'(t) \times \vec{r}\,''(t)|}{|\vec{r}\,'(t)|^3}$$

$$\vec{r}\,''(t) = \langle -2\cos t, -2\sin t, 0 \rangle$$

$$\vec{r}\,'(t) \times \vec{r}\,''(t) = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ -2\sin t & 2\cos t & 1 \\ -2\cos t & -2\sin t & 0 \end{vmatrix} = +2\sin t\,\vec{\imath} - 2\cos t\,\vec{\jmath} + 4(\underbrace{\sin^2 t + \cos^2 t}_{1})\vec{k}$$

$$|\vec{r}\,'(t) \times \vec{r}\,''(t)| = \sqrt{4(\underbrace{\sin^2 t + \cos^2 t}_{1}) + 16}$$

$$\kappa(t) = \frac{\sqrt{20}}{\sqrt{5}^3} = \frac{2}{5}.$$

(8) (a) Find the equation of the tangent plane to the surface

$$z = 2x^2 - 3y^2 + 2y - x$$

at the point $(-1, 2, -5)$.

Tangent plane by linearization

$$L(x,y) = -5 + \underbrace{\frac{\partial z(-1,2)}{\partial x}}_{-5}(x+1) + \underbrace{\frac{\partial z(-1,2)}{\partial y}}_{-10}(y-2)$$

$$\left.\begin{aligned} \frac{\partial z}{\partial x} &= 4x - 1 \\ \frac{\partial z}{\partial y} &= -6y + 2 \end{aligned}\right\} \Rightarrow L(x,y) = 10 - 5x - 10y$$

(b) Find the linearization $L(x, y)$ of the function

$$f(x, y) = \sin(x + 2y)$$

at the point $(\frac{\pi}{2}, \frac{\pi}{6})$ and use it to estimate $f(\frac{5\pi}{8}, \frac{\pi}{6})$.

$$f\left(\frac{\pi}{2}, \frac{\pi}{6}\right) = \sin\Big(\underbrace{\frac{\pi}{2} + \frac{\pi}{3}}_{\frac{5\pi}{6}}\Big) = \sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$$

$$f_x(x,y) = \cos(x + 2y)$$

$$f_x\left(\frac{\pi}{2}, \frac{\pi}{6}\right) = \cos\left(\frac{5\pi}{6}\right) = -\frac{\sqrt{3}}{2}$$

$$f_y(x,y) = 2\cos(x + 2y)$$

$$f_y\left(\frac{\pi}{2}, \frac{\pi}{6}\right) = -\sqrt{3}$$

$$\Rightarrow \quad L(x,y) = \frac{1}{2} - \frac{\sqrt{3}}{2}\left(x - \frac{\pi}{2}\right) - \sqrt{3}\left(y - \frac{\pi}{6}\right)$$

$$f\left(\frac{5\pi}{8}, \frac{\pi}{6}\right) \approx L\left(\frac{5\pi}{8}, \frac{\pi}{6}\right) = \frac{1}{2} - \frac{\sqrt{3}}{2}\Big(\underbrace{\frac{5\pi}{8} - \frac{4\pi}{8}}_{\frac{\pi}{8}}\Big)$$

$$= \frac{1}{2} \; 8 \; \frac{\sqrt{3}}{16}\pi .$$