# Practice Exam # 2
# Math 1451

**Spring 2015**
**Section ___________**

**Name: ________________________________ Last 4 digits of student ID: _________**

**The use of calculators or cell phones is not allowed in this exam. Show all the required work to obtain full credit. If running out of space, you may use the back sides of the pages or the empty page at the end. The duration of this exam is 80 minutes.**

(1) (a) Find the maximum rate of change of $f(x, y) = \sqrt{x^2 + 2y^2}$ at the point $(1, 1)$ and find a unit vector $\vec{u}$ in the direction in which it occurs.

(b) Using the gradient function or otherwise find the equation of the tangent plane to the ellipsoid $\frac{x^2}{2} + \frac{y^2}{4} + z^2 = \frac{5}{2}$ at the point $(1, 2, 1)$.

(2) Find the maximum and minimum values of $f(x, y) = 1 + x^2 - x + y^2 - y$ on the triangle with vertices $(0, 0)$, $(0, 1)$ and $(2, 1)$.

(3) (a) Is the following statement true or false? If false give a counterexample.
**If** $\frac{\partial f}{\partial x}(x_0, y_0) = \frac{\partial f}{\partial y}(x_0, y_0) = 0$ **at** $(x_0, y_0)$ **then** $f(x, y)$ **has a local maximum or a local minimum at** $(x_0, y_0)$.

(b) Is the following statement true or false? If false give a counterexample.
**If** $f(x, y)$ **has a local maximum or a local minimum at** $(x_0, y_0)$ **and** $\frac{\partial f}{\partial x}(x_0, y_0)$ **and** $\frac{\partial f}{\partial y}(x_0, y_0)$ **both exist at** $(x_0, y_0)$ **then** $\frac{\partial f}{\partial x}(x_0, y_0) = \frac{\partial f}{\partial y}(x_0, y_0) = 0$.

(4) A thin plate has the density $\rho(x, y) = xy^2$ and the shape $D = \{(x, y) : 0 \le y \le 3x, 0 \le x \le 1\}$.

(a) Find the mass $m$ of the plate

(b) Find the coordinates $\overline{x}$ and $\overline{y}$ of the plate's center of mass.

(5) Find the volume $V$ of the solid which is inside the sphere $x^2 + y^2 + z^2 = a^2$ and outside the cylinder $x^2 + y^2 = b^2$ where $a > b > 0$. *Hint: use polar or cylindrical coordinates.*

(6) Find

$$\iiint_E (x^2 + y^2 + z^2)^2 \, dV$$

where E is the solid described by

$$\{(x, y, z) : x \geq 0, y \geq 0, z \geq 0 \text{ and } 1 \leq x^2 + y^2 + z^2 \leq 4\}$$

*Hint: use spherical coordinates.*

(7) Find the integral of the function $f(x, y) = \frac{1}{x^2 + y^2 + 2}$ over the disc $D = \{(x, y) : 0 \leq x^2 + y^2 \leq A\}$ for a fixed value of $A \geq 0$. (Your answer will involve the constant $A$).

(8) An airline requires that the total outside dimensions (length+width+height) of a checked bag not exceed 90 inches. Suppose you want to check a bag whose height equals twice its width. What is the largest volume $V$ (in cubic inches) of a bag of this shape that you can check on a flight of this airline?

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1) a) Max rate of chg of $f(x,y) = \sqrt{x^2+2y^2}$ at $(1,1)$.

Compute $\max_{\|\vec{u}\|=1} \| D_{\vec{u}} f(1,1) \|$

$$= \| \vec{\nabla} f(1,1) \|$$

$$\vec{\nabla} f(x,y) = \frac{1}{2\sqrt{x^2+2y^2}} \langle 2x, 4y \rangle$$

$$= \frac{1}{\sqrt{x^2+2y^2}} \langle x, 2y \rangle$$

$$\Rightarrow \| \vec{\nabla} f(1,1) \| = \frac{1}{\sqrt{3}} \sqrt{1+2^2} = \sqrt{\frac{5}{3}}$$

Direction

$$\vec{u} = \frac{\langle 1, 2 \rangle}{\sqrt{1+2^2}} = \langle \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \rangle .$$

b) Constraint $h(x,y,z) = \frac{5}{2}$

with $h(x,y,z) = \frac{x^2}{2} + \frac{y^2}{4} + z^2$.

$$\vec{\nabla} h = \langle x, \frac{y}{2}, 2z \rangle$$

Equation for tangent plane at $(1,2,1)$

$$\vec{\nabla} h(1,2,1) = \langle 1, 1, 2 \rangle$$

$$(1)x + (1)y + (2)z = d$$

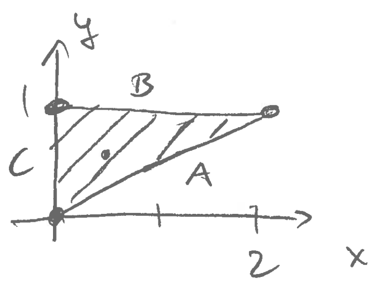
plug in $(1,2,1)$ gives $d = 1 + 2 + 2 = 5$

so

$$x + y + 2z = 5.$$

2) $f(x,y) = 1 + x^2 - x + y^2 - y$

find max/min on triangle




Crit. pts. $\vec{\nabla} f(x,y) = \langle 2x-1, 2y-1 \rangle$

$$\langle 2x-1, 2y-1 \rangle = \langle 0, 0 \rangle \Rightarrow x = \tfrac{1}{2}, y = \tfrac{1}{2}$$

$$f(\tfrac{1}{2}, \tfrac{1}{2}) = 1 + \tfrac{1}{4} - \tfrac{1}{2} + \tfrac{1}{4} - \tfrac{1}{2}$$

$$= \tfrac{1}{2}.$$

Boundaries:

A: $y = \frac{1}{2}x, \quad 0 \le x \le 2$

$$g(x) = 1 + x^2 - x + (\tfrac{1}{2}x)^2 - (\tfrac{1}{2}x)$$

$$= 1 + \tfrac{5}{4}x^2 - \tfrac{3}{2}x$$

CP $\quad g'(x) = 0 \Rightarrow \frac{5}{\not 2}x - \frac{3}{\not 2} = 0$

$$x = \tfrac{3}{5}, \quad y = \tfrac{3}{10}$$

$$\Rightarrow f(\tfrac{3}{5}, \tfrac{3}{10}) = 1 + \tfrac{9}{25} - \tfrac{3}{5} + \tfrac{9}{100} - \tfrac{3}{10}$$

$$= 1 - \tfrac{9}{10} + \tfrac{45}{100} = \tfrac{55}{100}$$

B: $y = 1, \quad 0 \le x \le 2$

$$g(x) = 1 + x^2 - x + \not 1 - \not 1$$

$$= 1 + (x - \tfrac{1}{2})^2 - \tfrac{1}{4}$$

$\Rightarrow$ min at $x = \frac{1}{2}$,

$$g(\tfrac{1}{2}) = f(\tfrac{1}{2}, 1) = \tfrac{1}{2} + \tfrac{1}{4} = \tfrac{3}{4}$$

C: $x = 0, \quad 0 \le y \le 1$

$$g(y) = 1 + y^2 - y$$

$\Rightarrow$ min of $g(y)$ at $y = \frac{1}{2}$,

$$g(\tfrac{1}{2}) = \tfrac{3}{4}$$

Vertices $f(0,0) = 1$

$f(0,1) = 1 + 1 - 1 = 1$

$f(2,1) = 1 + 4 - 2 + 1 - 1 = 3$

Global max is $f(2,1) = 3$,

global min is $f(\frac{1}{2}, \frac{1}{2}) = \frac{1}{2}$.

3) a) Let $x_0, y_0 = 0$ and $f(x,y) = x^2 - y^2$

then $\frac{\partial f}{\partial x}(0,0) = 0$, $\frac{\partial f}{\partial y}(0,0) = 0$

but no local max or min. False.

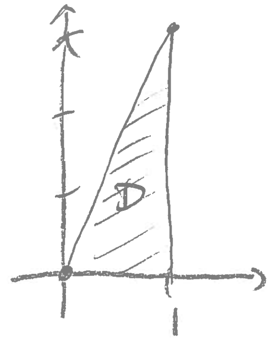
b) If $f$ has local max or min at $(x_0, y_0)$

then $\frac{\partial f}{\partial x}(x_0, y_0) = \frac{\partial f}{\partial y}(x_0, y_0) = 0$ if

they exist. True.

4) $\rho(x,y) = xy^2$

$D = \{(x,y) : 0 \le y \le 3x, \; 0 \le x \le 1\}$




$$m = \int_0^1 \int_0^{3x} xy^2 \, dy \, dx$$

$$= \int_0^1 x \left[\frac{1}{3} y^3\right]_0^{3x} dx$$

$$= \int_0^1 9x^3 (x) \, dx$$

$$= \frac{9}{5} \cancel{(1)^5} .$$

$$\bar{x} = \frac{1}{m} \int_0^1 \int_0^{3x} x^2 y^2 \, dy \, dx$$

$$= \frac{1}{m} \int_0^1 x^2 \left[ \frac{1}{3} y^3 \right]_0^{3x} dx$$

$$= \frac{1}{m} \int_0^1 9 x^5 \, dx = \frac{1}{m} \frac{9}{6}$$

$$= \frac{5}{6}$$

$$\bar{y} = \frac{1}{m} \int_0^1 \int_0^{3x} x y^3 \, dy \, dx$$

$$= \frac{1}{m} \int_0^1 x \left[ \frac{1}{4} y^4 \right]_0^{3x} dx$$

$$= \frac{1}{m} \frac{81}{4} \underbrace{\int_0^1 x^5 \, dx}_{\frac{1}{6}}$$

$$= \frac{5}{\cancel{9}} \frac{\cancel{81}\,9}{24} = \frac{45}{24} = \frac{15}{8}.$$

5) Cyl. coords

$$E = \{(x,y,z) : x^2+y^2+z^2 \le a^2, \; x^2+y^2 \ge b^2\}$$

$$b^2 \le x^2+y^2 = r^2 \le a^2$$

$$V = \iiint_E dV$$

$$= \int_0^{2\pi} \int_b^a \int_{-\sqrt{a^2-r^2}}^{\sqrt{a^2-r^2}} dz \, r dr \, d\theta$$

$$= 2\pi \int_b^a 2\sqrt{a^2-r^2} \, r dr$$

$$= 2\pi \left[ -\frac{2}{3}(a^2-r^2)^{3/2} \right]_b^a$$

$$= \frac{4\pi}{3} (a^2-b^2)^{3/2}$$

6) By symmetry, if

$$B = \{(x,y,z) : 1 \le x^2+y^2+z^2 \le \underbrace{4}\}$$

then $\rho^2 = 4 \Rightarrow \rho = 2$

$$\iiint_E (x^2+y^2+z^2)^2 \, dV$$

$$= \frac{1}{8} \iiint_B (\underbrace{x^2+y^2+z^2}_{\rho^2})^2 \, dV$$

$$= \frac{1}{8} \int_0^{2\pi} \int_0^{\pi} \int_1^2 \rho^4 \rho^2 \, d\rho \, \sin\phi \, d\phi \, d\theta$$

$$= \frac{1}{8} (2\pi) \underbrace{\int_0^{\pi} \sin\phi \, d\phi}_{2} \underbrace{\int_1^2 \rho^6 \, d\rho}_{\frac{1}{7}\rho^7 \Big|_1^2}$$

$$= \frac{\pi}{2} \left( \frac{128}{7} - \frac{1}{7} \right) = \frac{127\pi}{14}.$$

7) $\iint_D f(x,y)\,dx\,dy$

$$= \int_0^{2\pi} \int_0^{\sqrt{A}} \frac{1}{r^2+2}\, r\,dr\,d\theta$$

$$= \int_0^{2\pi} d\theta \;\; \frac{1}{2} \ln\left[r^2+2\right] \Big|_0^{\sqrt{A}}$$

$$= \cancel{2}\pi \left(\ln(A+2) - \ln 2\right)$$

$$= \pi \ln\left(\frac{A}{2}+1\right).$$

8) $l + w + h = 90$

$2w = h$

$\Rightarrow \quad l + 3w = 90 \qquad (*)$

$$V = (l)(w)(h) = (l)(w)(2w)$$

$$= 2lw^2$$

$$h(l,w) = l + 3w$$

$$\Rightarrow \vec{\nabla} h = \langle 1, 3 \rangle$$

$$\vec{\nabla} V = \langle 2w^2, 4lw \rangle$$

Lagrange $\Rightarrow$

$$1\lambda = 2w^2$$

$$3\lambda = 4lw$$

$$\Rightarrow \quad \frac{1}{3} = \frac{1}{2}\frac{w}{l} \quad \Rightarrow \quad 2l = 3w$$

in (*) :

$$\frac{3}{2}w + 3w = 90$$

$$\parallel$$

$$\frac{9}{2}w$$

$$\Rightarrow \quad w = 20 \quad \Rightarrow \quad h = 40$$

$$\Rightarrow \quad l = \frac{3}{2}w = 30$$

$$V = (20)(30)(40) = 24{,}000 \ (in^3).$$