Practice Final Exam
Math 1451

Spring 2015
Section ____________

Name: ______________________________ **Last 4 digits of student ID:** ________

The use of calculators or cell phones is not allowed in this exam. Show all the required work to obtain full credit. If running out of space, you may use the back sides of the pages or the empty page at the end. The duration of this exam is 150 minutes.

(1) Find parametric vector equations of the tangent line to the curve given by $\vec{r}(t) = \langle e^{-t}\cos t, e^{-t}\sin t, e^{-t}\rangle$ at $(1, 0, 1)$.

(2) Calculate the following line integrals:

(a) $\int_C f ds$ where $f(x, y) = 4xy$ and C is the straight line segment between $(-1, -1)$ and $(2, 1)$.

(b) $\int_C \vec{F} \cdot d\vec{r}$ where $\vec{F}(x, y) = \langle -x, y^2 \rangle$ and C is the part of the curve $y = x^2$ starting at $(1, 1)$ and ending at $(2, 4)$.

(c) $\int_C \vec{F} \cdot d\vec{r}$ where $\vec{F} = \langle y^2, x \rangle$ and C is the boundary of the unit square in the plane with vertices $(0, 0)$, $(1, 0)$, $(1, 1)$ and $(0, 1)$ parametrized counter-clockwise.

(3) Let $\vec{F}(x,y,z) = \langle yz + e^{-y} - ye^{-x}, xz + e^{-x} - xe^{-y}, xy \rangle$.

(a) Find a function $f(x,y,z)$ such that $\nabla f = \vec{F}$.

(b) Compute $\int_C \vec{F} \cdot d\vec{r}$, where C is parametrized by $\vec{r}(t) = \langle 1, t, t^2 \rangle, 0 \leq t \leq 1$.

(4) (a) If $\vec{F}$ is a C^2-vector field on $\mathbb{R}^3$, does div(curl $\vec{F}$) $= 0$? Either prove the identity or give a counterexample.

(b) If f is a scalar function on $\mathbb{R}^3$, does div$(\nabla f) = 0$? Either prove or give a counterexample.

(5) Find the region which is enclosed by the curve described in polar coordinates by $r = 4+3\cos\theta$.

(6) Find the absolute maximum (value) of the function $f(x, y) = x^2 + y^2 + x^2y^4$, on $D = \{(x, y) : |x| \leq 1, |y| \leq 1\}$

(7) A package in the shape of a rectangular box can be mailed by the US Postal Service if the sum of its length l and girth g (the perimeter of a cross-section perpendicular to the length) is at most 108 in. Find the dimensions l, w, and h (length, width and height) of the package with largest volume that can be mailed.

(8) Use the transformation $x = u^2$, $y = v^2$, $z = w^2$ to find the volume of the region bounded by the surface $\sqrt{x} + \sqrt{y} + \sqrt{z} = 1$ and the coordinate planes.

(9) A particle starts at the point $(0, 0)$ and moves along the x axis to $(1, 0)$, then along that part of the circle $x^2 + y^2 = 1$ between $(1, 0)$ and $(0, 1)$ and then along the y axis to $(0, 0)$ again. Use Green's Theorem to find the work done on the particle by the force field

$$F(x, y) = (x, x^2 + xy)$$

by traversing this curve. *Hint: use polar coordinates.*

(10) Let S be the sphere

$$x^2 + y^2 + z^2 = 1$$

$\vec{n}$ the outward pointing unit normal to the sphere, and $\vec{F}$ the vector field

$$\vec{F}(x, y, z) = (\cos^2 x, \sin^2 y, z)$$

Find

$$\int_S \vec{F} \cdot \vec{n} dS \, .$$

(11) Use Stokes's Theorem to evaluate $\int_C \vec{F} \cdot d\vec{r}$, where $\vec{F}(x, y, z) = \langle xy, yz, zx \rangle$, and C is the triangle with vertices $(1, 0, 0), (0, 1, 0)$, and $(0, 0, 1)$, oriented counter-clockwise when viewed from above.

(12) If f is C^2 and a harmonic function on $\mathbb{R}^2$, that is, $\nabla \cdot \nabla f = f_{x,x} + f_{y,y} = 0$, show that the line integral $\int_C (f_y dx - f_x dy)$ is path independent.

(13) Is it true that if $\vec{r}(t)$ is a smooth curve in $\mathbb{R}^3$ between $\vec{r}(a)$ and $\vec{r}(b)$, then there exists a s in (a, b) such that $\vec{r}'(s) = (\vec{r}(b) - \vec{r}(a))/(b - a)$, similar to the mean value theorem of calculus in one dimension? Either prove or find a counterexample.

(14) Show that if a plane is given by the equation

$$\vec{r} \cdot \vec{n} = 0$$

with a fixed non-zero vector $\vec{n}$ and if a vector $\vec{w} = \langle p, q, u \rangle$ is not in this plane, then the vector $\vec{v} = \langle x, y, z \rangle$ in the plane whose distance to $\vec{w}$ is minimal satisfies the orthogonality property $(\vec{w} - \vec{v}) \cdot \vec{v} = 0$.

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(1) Find parametric vector ~~equations~~ equation of the tangent line to the curve given by $\vec{r}(t) = \langle e^{-t}\cos t, e^{-t}\sin t, e^{-t}\rangle$ at $(1, 0, 1)$.

$$\vec{r}\,'(t) = \langle -e^{-t}\cos t - e^{-t}\sin t,$$
$$-e^{-t}\sin t + e^{-t}\cos t,$$
$$-e^{-t} \rangle$$

$$e^{-t} = 1 \Rightarrow t = 0$$

$$\vec{r}\,'(0) = \langle -1 - 0, -0 + 1, -1 \rangle$$

Param. vector equation

$$\vec{\ell}(t) = \langle 1, 0, 1 \rangle + t\langle -1, 1, -1 \rangle.$$

(2) Calculate the following line integrals:

(a) $\int_C f ds$ where $f(x, y) = 4xy$ and C is the straight line segment between $(-1, -1)$ and $(2, 1)$.

Parametrize : $\vec{r}(t) = \langle -1, -1 \rangle + t(\langle 2, 1 \rangle - \langle -1, -1 \rangle$

$= \langle -1 + 3t, -1 + 2t \rangle, \quad 0 \le t \le$

$|\vec{r}'(t)| = \sqrt{3^2 + 2^2} = \sqrt{13}$

$$\int_C f ds = \int_0^1 4(-1+3t)(-1+2t)\sqrt{13}\, dt$$

$$= 4\sqrt{13} \int_0^1 (6t^2 - 5t + 1)\, dt$$

$$= 4\sqrt{13}\left(\frac{6}{3} 1^3 - \frac{5}{2} 1^2 + 1\right)$$

$$= 2\sqrt{13}$$

(b) $\int_C \vec{F} \cdot d\vec{r}$ where $\vec{F}(x,y) = \langle -x, y^2 \rangle$ and C is the part of the curve $y = \underbrace{x^2}_{\text{graph}}$ starting at $(1,1)$ and ending at $(2,4)$.

$$\int_C \vec{F} \cdot d\vec{r} = \int_1^2 \langle -x, \underbrace{(y(x))^2}_{x^4} \rangle \cdot \langle 1, \underbrace{y'(x)}_{2x} \rangle \, dx$$

$$= \int_1^2 (-x + 2x^5)\, dx$$

$$= \left[-\frac{1}{2}x^2 + \frac{2}{6}x^6 \right]_1^2 = -\frac{4}{2} + \frac{1}{2} + \frac{128}{6} -$$

$$= \frac{117}{6}$$

(c) $\int_C \vec{F} \cdot d\vec{r}$ where $\vec{F} = \langle y^2, x \rangle$ and C is the boundary of the unit square in the plane with vertices $(0,0)$, $(1,0)$, $(1,1)$ and $(0,1)$ parametrized counter-clockwise. Green !

$$\int_C \vec{F} \cdot d\vec{r} = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

$$= \int_0^1 \int_0^1 (1 - 2y)\, dy\, dx$$

$$= \int_0^1 (1 - 2y)\, dy = 1 - \frac{2}{2} = 0$$

(3) Let $\vec{F}(x,y,z) = \langle yz + e^{-y} - ye^{-x}, xz + e^{-x} - xe^{-y}, xy\rangle$.

(a) Find a function $f(x,y,z)$ such that $\nabla f = \vec{F}$.

$$\frac{\partial f}{\partial x} = yz + e^{-y} - ye^{-x}$$

$$\Rightarrow \quad f(x,y,z) = yzx + e^{-y}x + ye^{-x} + g(y,z)$$

$$\frac{\partial f}{\partial y} = xz - xe^{-y} + e^{-x} + \frac{\partial g(y,z)}{\partial y}$$

$$\Rightarrow \quad \frac{\partial g}{\partial y} = 0 \quad \Rightarrow \quad g(y,z) = h(z)$$

$$\frac{\partial f}{\partial z} = xy + \underbrace{h'(z)}_{0} \quad \Rightarrow \quad f(x,y,z) = xyz + xe^{-y} + ye^{-x} + \underbrace{C}_{}$$

can choose $C = 0$

(b) Compute $\int_C \vec{F} \cdot d\vec{r}$, where C is parametrized by $\vec{r}(t) = \langle 1, t, t^2\rangle, 0 \le t \le 1$.

$$\int_C \vec{F} \cdot d\vec{r} = f(\vec{r}(1)) - f(\vec{r}(0))$$

$$= (1 + e^{-1} + e^{-1}) - (1)$$

$$= \cancel{1} + \frac{2}{e}$$

(4) (a) If $\vec{F}$ is a C^2-vector field on $\mathbb{R}^3$, does div(curl $\vec{F}$) $= 0$? Either prove the identity or give a counterexample.

$$\left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \underbrace{\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}}$$

$$\langle R_y - Q_z \,,\; -R_x + P_z \,,\; Q_x - P_y \rangle$$

$$= \cancel{R_{yx}} - \cancel{Q_{zx}} - \cancel{R_{xy}} + \cancel{P_{zy}} + Q_{xz} - P_{yz}$$

Clairaut $= 0$

(b) If f is a scalar function on $\mathbb{R}^3$, does div$(\nabla f) = 0$? Either prove or give a counterexample.

No, take $f(x,y,z) = x^2 + y^2 + z^2$.

$$\vec{\nabla} \cdot \vec{\nabla} f = \vec{\nabla} \cdot \langle 2x, 2y, 2z \rangle$$

$$= \frac{\partial}{\partial x}(2x) + \frac{\partial}{\partial y}(2y) + \frac{\partial}{\partial z}(2z)$$

$$= 2 + 2 + 2 = 6 \neq 0$$

(5) Find the area of the region which is enclosed by the curve described in polar coordinates by $r = 4 + 3\cos\theta$.

$r(\theta) \geq 4 - 3 = 1$

range for θ is $[0, 2\pi]$

$$\text{Area} = \int_0^{2\pi} \underbrace{\int_0^{4+3\cos\theta} r\,dr}_{\left[\frac{1}{2} r^2\right]_0^{4+3\cos\theta}} \, d\theta$$

$$= \int_0^{2\pi} \frac{1}{2} \underbrace{(4 + 3\cos\theta)^2}_{16 + 24\cos\theta + 9 \underbrace{\cos^2\theta}_{\frac{1}{2} + \frac{1}{2}\cos(2\theta)}} \, d\theta$$

$$= 8(2\pi) + 12 \underbrace{\int_0^{2\pi} \cos\theta \, d\theta}_{0} + \frac{9}{4}(2\pi)$$

$$= \frac{41}{2}\pi$$

(6) Find the absolute maximum (value) of the function $f(x, y) = x^2 + y^2 + x^2y^4$, on $D = \{(x, y) : |x| \leq 1, |y| \leq 1\}$

Crit pts. $\vec{\nabla} f = \langle 2x + 2xy^4, 2y + 4x^2y^3 \rangle$

$= \langle 0, 0 \rangle$

$\underbrace{2x(1+y^4)}_{\geq 1} = 0 \Rightarrow x = 0$

$\Rightarrow 2y + 0 = 0$

$\Rightarrow y = 0$

$f(0,0) = 0$

Boundaries :

A : $x = 1, \quad -1 \leq y \leq 1,$

$f(1, y) = 1 + y^2 + y^4 = g(y)$

$g'(y) = 2y + 4y^3 = 0$

$2y(1 + \underbrace{2y^2}_{\geq 0}) = 0$

$\Rightarrow y = 0$

$f(1, 0) = 1 + 0$

B: $x = -1, \quad -1 \leq y \leq 1$

$f(-1, y) = (-1)^2 + y^2 + y^4$

(same as A)

C: $y = 1, \quad -1 \leq x \leq 1$

$f(x, 1) = x^2 + 1 + x^2$

$= 1 + 2x^2 = g(x)$

$g'(x) = 4x = 0 \Rightarrow x = 0$

$f(0, 1) = 1$

D: $y = -1, \quad -1 \leq x \leq 1$

$f(x, -1) = 1 + 2x^2$ (same as C)

Corner points: $f(1,1) = 3$

$f(-1,1) = 3$

$f(-1,-1) = 3$

$f(1,-1) = 3$

$\Rightarrow$ max value is 3, attained at corners.

(7) A package in the shape of a rectangular box can be mailed by the US Postal Service if the sum of its length l and girth g (the perimeter of a cross-section perpendicular to the length) is at most 108 in. Find the dimensions l, w, and h (length, width and height) of the package with largest volume that can be mailed.

$$V = \ell w h$$

$$H = \ell + g = \ell + 2w + 2h = 108$$

$$\vec{\nabla} V = \langle wh, \ \ell h, \ \ell w \rangle$$

$$\vec{\nabla} H = \langle 1, \ 2, \ 2 \rangle$$

$$\vec{\nabla} V = \lambda \vec{\nabla} H$$

$$\Rightarrow \quad wh = \lambda \qquad (1)$$

$$\ell h = 2\lambda \qquad (2)$$

$$\ell w = 2\lambda \qquad (3)$$

boundary: $w = 0$ or $h = 0$ gives $V = 0$ no max

$$\Rightarrow \quad \frac{(1)}{(2)} : \quad \frac{w}{\ell} = \frac{1}{2} \quad \Rightarrow \quad w = \frac{1}{2}\ell$$

$$\frac{(1)}{(3)} : \quad \frac{h}{\ell} = \frac{1}{2} \quad \Rightarrow \quad h = \frac{1}{2}\ell$$

$$\Rightarrow \quad \ell + 2\left(\tfrac{1}{2}\ell\right) + 2\left(\tfrac{1}{2}\ell\right) = 3\ell = 108$$

$$\ell = 36$$

$$\Rightarrow \quad \ell = 36 \quad w = 18, \quad h = 18$$

(8) Use the transformation $x = u^2$, $y = v^2$, $z = w^2$ to find the volume of the region bounded by the surface $\sqrt{x} + \sqrt{y} + \sqrt{z} = 1$ and the coordinate planes.

$$\iiint_E dV = \iiint_R \left| \frac{\partial(x,y,z)}{\partial(u,v,w)} \right| dV$$

$$= \iiint_R (2u)(2v)(2w)\, dV$$

with $R = \{(u,v,w) : u + v + w \le 1,\ u, v, w \ge 0\}$

$$\iiint_E dV = 8 \int_0^1 u \int_0^{1-u} v \underbrace{\int_0^{1-u-v} w\, dw}_{\frac{1}{2}(1-u-v)^2}\, dv\, du$$

$$= 4 \int_0^1 u \int_0^{1-u} v \underbrace{(1-u-v)^2}_{(1-u)^2 - 2(1-u)v + v^2} dv \quad du$$

$$= 4 \int_0^1 u \underbrace{\left[\frac{1}{2}(1-u)^4 - \frac{2}{3}(1-u)^4 + \frac{1}{4}(1-u)^4 \right]}_{\frac{1}{12}(1-u)^4} d$$

$$= \frac{1}{3} \int_0^1 u(1-u)^4 \, du$$

$u' = 1 - u$

$du' = -du$

$$= \frac{1}{3} \int_0^1 (1-u')(u')^4 \, du'$$

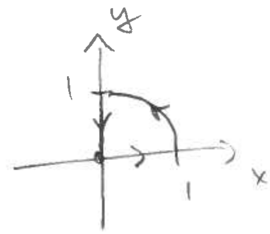
$$= \frac{1}{3}\left(\frac{1}{5} - \frac{1}{6}\right) = \frac{1}{90}$$

(9) A particle starts at the point $(0,0)$ and moves along the x axis to $(1,0)$, then along that part of the circle $x^2+y^2=1$ between $(1,0)$ and $(0,1)$ and then along the y axis to $(0,0)$ again. Use Green's Theorem to find the work done on the particle by the force field

$$F(x,y) = (x, x^2+xy)$$

by traversing this curve. *Hint: use polar coordinates.*

Sketch :



$$\int_C \vec{F}\cdot d\vec{r} = \iint_D (Q_x - P_y)\, dA$$

$$= \iint_D (2x + y - 0)\, dA$$

$$= \int_0^{\pi/2} \int_0^1 (2r\cos\theta + r\sin\theta)\, r\, dr\, d\theta$$

$$= \int_0^{\pi/2} \left[\frac{2}{3}\cos\theta + \frac{1}{3}\sin\theta \right] d\theta$$

$$= \left[+\frac{2}{3}\sin\theta - \frac{1}{3}\cos\theta \right]_0^{\pi/2}$$

$$= \frac{2}{3} + \frac{1}{3} = 1$$

(10) Let S be the sphere

$$x^2 + y^2 + z^2 = 1$$

$\vec{n}$ the outward pointing unit normal to the sphere, and $\vec{F}$ the vector field

$$\vec{F}(x, y, z) = (\cos^2 x, \sin^2 y, z)$$

Find

$$\int_S \vec{F} \cdot \vec{n} dS .$$

Gauss :

$$\iint_S \vec{F} \cdot \vec{n} \, dS$$

$$= \iiint_E \underbrace{\vec{\nabla} \cdot \vec{F}}_{2(\cos x)(-\sin x) + 2\sin y \cos y + 1} \, dV$$

$$= \iint_D \int_{-\sqrt{1-y^2-z^2}}^{\sqrt{1-y^2-z^2}} \underbrace{2(\cos x)(-\sin x)}_{\text{odd}} \, dx \, dA$$

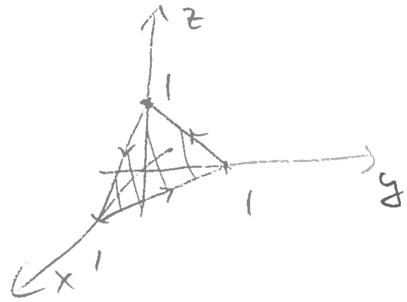
$$+ \iint_D \int_{-\sqrt{1-x^2-z^2}}^{\sqrt{1-x^2-z^2}} \underbrace{2(\sin y)(\cos y)}_{\text{odd}} \, dy \, dA$$

$$+ \iiint_E (1) \, dV$$

$$= 0 + 0 + \text{vol}(\text{ball}) = \frac{4}{3}\pi .$$

(11) Use Stokes's Theorem to evaluate $\int_C \vec{F} \cdot d\vec{r}$, where $\vec{F}(x, y, z) = \langle xy, yz, zx \rangle$, and C is the triangle with vertices $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$, oriented counter- clockwise when viewed from above.

Sketch :



S : graph of $z = 1 - x - y$, $D = \{(x,y) : x + y \leq 1, \ x, y \geq 0\}$

$\Rightarrow \vec{n} = \frac{1}{\sqrt{3}} \langle 1, 1, 1 \rangle$

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & yz & zx \end{vmatrix} = -y\vec{i} - z\vec{j} + -x\vec{k}$$

$$= \langle -y, -z, -x \rangle$$

$$\int_C \vec{F} \cdot d\vec{r} = \iint_S \vec{\nabla} \times \vec{F} \cdot \vec{n} \, dS$$

$$= \iint_S \frac{1}{\sqrt{3}} \underbrace{(-y - z - x)}_{-1} \, dS$$

$$= -\frac{1}{\sqrt{3}} \text{area}(S)$$

$$\underbrace{\text{area}(S)}_{\text{triangle}} = \frac{1}{2}\,\overbrace{\left|\langle 0,1,0\rangle - \langle 1,0,0\rangle\right|} * \underbrace{\left|\langle 0,0,1\rangle - \langle \tfrac{1}{2},\tfrac{1}{2},0\rangle\right|}_{\text{height}}$$

$$= \frac{1}{2}\sqrt{2}\sqrt{1+\frac{1}{4}+\frac{1}{4}} = \frac{1}{2}\sqrt{\cancel{2}}\sqrt{\frac{3}{\cancel{2}}}$$

$$\Rightarrow \int_C \vec{F}\cdot d\vec{r} = -\frac{1}{\cancel{\sqrt{3}}}\left(\frac{1}{2}\cancel{\sqrt{3}}\right) = -\frac{1}{2}.$$

(12) If f is C^2 and a harmonic function on $\mathbb{R}^2$, that is, $\nabla \cdot \nabla f = f_{x,x} + f_{y,y} = 0$, show that the line integral $\int_C (f_y dx \cancel{+} f_x dy)$ is path independent.

By $\vec{F} = \langle f_y, -f_x \rangle$

$$\int_C (f_y\, dx - f_x\, dy)$$

$$= \int_C \vec{F} \cdot d\vec{r}$$

We test $\dfrac{\partial Q}{\partial x} = -f_{xx}$, $\dfrac{\partial P}{\partial y} = f_{yy}$

since $f_{x,x} + f_{y,y} = 0$, $\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$

and $\vec{F}$ is grad. vector field

Thus, $\int_C \vec{F} \cdot d\vec{r}$ is path indep.

(13) Is it true that if $\vec{r}(t)$ is a smooth curve in $\mathbb{R}^3$ between $\vec{r}(a)$ and $\vec{r}(b)$, then there exists a s in (a, b) such that $\vec{r}'(s) = (\vec{r}(b) - \vec{r}(a))/(b - a)$, similar to the mean value theorem of calculus in one dimension? Either prove or find a counterexample.

Not true. Pick $\vec{r}(b) = \vec{r}(a) = \vec{0}$,

let $\vec{r}(t) = \langle \cos t - 1, \sin t, 0 \rangle$

then $\vec{r}'(t) = \langle -\sin t, \cos t, 0 \rangle$, $0 \leq t \leq ?$

so $\vec{r}'(t) = \vec{0} \Rightarrow \sin t = \cos t = 0$

<u>no</u> such t exists

but $\vec{r}(2\pi) = \vec{r}(0) = \langle 0, 0, 0 \rangle$

$\vec{r}(2\pi) - \vec{r}(0) = \vec{0}$.

(14) Consider a plane in $\mathbb{R}^3$ given by the equation

$$\vec{r} \cdot \vec{n} = 0$$

with a fixed non-zero vector $\vec{n}$. Let $\vec{w} = \langle p, q, r \rangle$ be a vector that is not in this plane, and $\vec{v} = \langle x, y, z \rangle$ be the vector in the plane whose distance to $\vec{w}$ is minimal. Explain that $\vec{w}$ and $\vec{v}$ satisfy the orthogonality property $(\vec{w} - \vec{v}) \cdot \vec{v} = 0$. Hint: Lagrange multipliers.

Constraint function

$$f(\vec{r}) = \vec{r} \cdot \vec{n} = 0$$

$$\Rightarrow \quad \vec{\nabla} f = \vec{n}$$

Squared Distance to $\vec{\omega}$,

$$D^2(\vec{r}, \vec{\omega}) = |\vec{r} - \vec{\omega}|^2$$

$$\vec{\nabla} D^2(\vec{r}, \vec{\omega}) = (+2)\langle r_x - p, r_y - q, r_z - r \rangle$$

$$= +2(\vec{r} - \vec{\omega})$$

At minimum,

$$\vec{\nabla} D^2 = \lambda \vec{\nabla} f, \quad \Rightarrow \quad +2(\vec{v} - \vec{\omega}) = \lambda \vec{n}$$

with $\vec{v}$ in plane,

$$\frac{\lambda}{2} \vec{n} \cdot \vec{v} = (\vec{v} - \vec{\omega}) \cdot \vec{v} = 0.$$