## 4.2 Null space, column space and linear transfns

**Def** The *null space* of an $m \times n$ matrix $A$, written Nul $A$, is the set of all solutions to the equation $A\vec{x} = \vec{0}$, so

$$\text{Nul } A = \{ \vec{x} \in \mathbb{R}^n : A\vec{x} = \vec{0} \}$$

**Thm** The null space of any $m \times n$ matrix $A$ is a subspace of $\mathbb{R}^n$. Equivalently the solutions to a homogeneous linear system, written in vector form, form a subspace of $\mathbb{R}^n$.

Why? Check:

a) $A\vec{0} = \vec{0}$, so $\vec{0} \in \text{Nul}(A)$

b) If $A\vec{u} = \vec{0}$ and $A\vec{v} = \vec{0}$, then $A(\vec{u} + \vec{v}) = \vec{0}$.

If $\vec{u}, \vec{v} \in \text{Nul}(A)$, then so is $\vec{u} + \vec{v}$.

c) If $A\vec{u} = \vec{0}$, $\overset{c \in \mathbb{R}}{\vee}$ then $A(c\vec{u}) = c(A\vec{u}) = \vec{0}$.

Thus, if $\vec{u} \in \text{Nul}(A)$, so is $c\vec{u}$ for any $c$.

Q.: How can we describe Nul(A) in terms of A?

Not clear - requires solving a system.

Recall: Solution in parametric vector form.

Let

$$A = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix}$$

find all $\vec{x}$ s.th. $A\vec{x} = \vec{0}$ in vector form.

Aug matrix

$$[A\ \vec{0}] \sim \begin{bmatrix} 1 & -2 & 2 & 3 & -1 & 0 \\ 0 & 0 & 5 & 10 & -10 & 0 \\ 0 & 0 & 1 & 2 & -2 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -2 & 2 & 3 & -1 & 0 \\ 0 & 0 & 1 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -2 & 0 & -1 & 3 & 0 \\ 0 & 0 & 1 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

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$$x_1 - 2x_2 \qquad - x_4 + 3x_5 = 0$$

$$x_3 + 2x_4 - 2x_5 = 0$$

$x_2, x_4, x_5$: free

$$x_1 = 2x_2 + x_4 - 3x_5$$

$$x_3 = -2x_4 + 2x_5$$

$x_2, x_4, x_5$ : free

$$\vec{x} = \begin{bmatrix} 2x_2 + x_4 - 3x_5 \\ x_2 \\ -2x_4 + 2x_5 \\ x_4 \\ x_5 \end{bmatrix} = x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix}$$

$$= x_2 \vec{u} + x_4 \vec{v} + x_5 \vec{w}$$

We have found a spanning set,

$$Nul(A) = span\{\vec{u}, \vec{v}, \vec{w}\} \quad !$$

Note: This set is <u>automatically</u> linearly independent;

$$x_2 \begin{bmatrix} 2 \\ \boxed{1} \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ -2 \\ \boxed{1} \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -3 \\ 0 \\ 2 \\ 0 \\ \boxed{1} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

implies $x_2 = 0$, $x_4 = 0$, $x_5 = 0$

If $\mathrm{Nul}(A) \neq \{\vec{0}\}$, then number of vectors for our spanning set is the number of free variables in $A\vec{x} = \vec{0}$.

Column space

Def The column space of an $m \times n$ matrix $A$ written $\mathrm{Col}(A)$, is the set of all linear combinations of the columns of $A$, so if $A = [\vec{a}_1\ \vec{a}_2\ \cdots\ \vec{a}_n]$,

$$\mathrm{Col}(A) = \mathrm{span}\{\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n\}$$

Thm The column space of an $m \times n$ matrix $A$ is a subspace of $\mathbb{R}^m$.

Example: For which $A$ is $W = \mathrm{Col}\,A$, with

$$W = \left\{ \begin{bmatrix} x - 2y \\ 3y \\ x + y \end{bmatrix} : x, y \in \mathbb{R} \right\} ?$$

We have

$$\begin{bmatrix} x - 2y \\ 3y \\ x + y \end{bmatrix} = x \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + y \begin{bmatrix} -2 \\ 3 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -2 \\ 0 & 3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

so $A = \begin{bmatrix} 1 & -2 \\ 0 & 3 \\ 1 & 1 \end{bmatrix}$.

We recall from Ch. 1:

The column space of a $m \times n$ matrix $A$ is all of $\mathbb{R}^m$ if and only if $A\vec{x} = \vec{b}$ ... is consistent for each $\vec{b} \in \mathbb{R}^m$.

Comparison between $Col(A)$, $Nul(A)$

Let

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 7 \\ 3 & 6 & 10 \\ 0 & 0 & 1 \end{bmatrix}$$

then $Col\, A$ is a subspace of $\mathbb{R}^4$.

$Nul\, A$ is — " — of $\mathbb{R}^3$.

$$Col\, A = \{ \vec{b} \in \mathbb{R}^4 : \vec{b} = x_1\vec{a}_1 + x_2\vec{a}_2 + x_3\vec{a}_3 \}.$$
$$= A\vec{x}$$

Find $Nul\, A$

$$[A\ \vec{0}] \sim \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$
$$\uparrow$$

$x_1 = -2x_2$

$x_2$ : free

$x_3 = 0$

$\vec{x} =$

$$\vec{x} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}.$$

By $\text{Col}(A) = \{ \vec{b} \in \mathbb{R}^m : A\vec{x} = \vec{b} \text{ for some } \vec{x} \in \mathbb{R}^n \}$

we see $\text{Col}(A)$ is the range of $T : \mathbb{R}^n \to \mathbb{R}^m$ with standard matrix $A$.

| Comparison | Nul A | Col A |
|---|---|---|
| Vectors | $\vec{v} \in \text{Nul } A$ satisfies $A\vec{v} = \vec{0}$ | $\vec{v} \in \text{Col } A$ satisfies $A\vec{x} = \vec{v}$ for some $\vec{x} \in \mathbb{R}^n$ |
| Subspace | Writing down spanning set requires effort | Writing down spanning set is easy |
| Test if some $\vec{v}$ is in subsp. | Easy: check $A\vec{v} = \vec{0}$ | Effort: solve $A\vec{x} = \vec{v}$ |
| "Extreme" cases | $\text{Nul } A = \{0\}$ if and only if $A\vec{x} = \vec{0}$ only has triv. solution, or: $T : \vec{x} \mapsto A\vec{x}$ is one-to-one | $\text{Col } A = \mathbb{R}^m$ if and only if $A\vec{x} = \vec{b}$ is always consistent, or $T : \vec{x} \leftrightarrow A\vec{x}$ is onto |