MATH 4331

Introduction to Real Analysis

Fall 2013

First name: ____________________ **Last name:** ______________________ **Points:**

Assignment 8, due Thursday, November 14, 10am

Please staple this cover page to your homework. When asked to prove something, make a careful step-by-step argument. You can quote anything we covered in class in support of your reasoning.

Problem 1

Let $f(x) = \frac{x}{2} + \frac{1}{x}$. Use some basic calculus to show that f maps $[1, 2]$ into $[1, 2]$, and use the mean value theorem to show that it is a contraction mapping. What is the value of the unique fixed point x_0? If you choose $x_1 = \frac{3}{2}$ as your starting value, estimate $|x_0 - x_n|$ for $n \in \mathbb{N}$.

Problem 2

Let $f(x) = x^2 - 5$. Show that f has a root x_0 somewhere in the interval $[2, 3]$. Calculate Newton's $g(x)$ and prove that g maps $[2, 3]$ into $[2, 3]$, with $g'(x) \leq \frac{1}{2}$ for $x \in [2, 3]$. Prove that if we perform Newton's method with $x_1 = 2$, then $|x_n - x_0| \leq \frac{1}{2^n}$.

Problem 3

Let $f(x) = x - \cos(x)$ so if x_0 is a root for f, then $\cos(x_0) = x_0$. Compute Newton's $g(x)$ and find concrete numbers a and b with $0 \leq a \leq b \leq 1$ such that g maps $[a, b]$ into $[a, b]$ and it is a contraction mapping. How does the Lipschitz constant of g compare with the one we had in class when we discussed the fixed point for $\cos x$?

Problem 4

Let $a, y_0 \in \mathbb{R}$. Solve the initial value problem $y'(x) = ay(x), y(0) = y_0$ on the interval $[0, \frac{1}{2a}]$ with the help of the contraction mapping theorem.

1. First show that Φ as defined in class is a contraction mapping on $C([0, \frac{1}{2a}])$.
2. Let $f_1(t) = y_0$ and define $f_{n+1} = \Phi(f_n)$ for $n \in \mathbb{N}$ as discussed in class. Compute f_2 and f_3. Can you guess f_n?