MATH 4331

Introduction to Real Analysis

Fall 2013

First name: ____________________ **Last name:** ______________________ **Points:**

Assignment 9, due Thursday, December 5, 10am

Please staple this cover page to your homework. When asked to prove something, make a careful step-by-step argument. You can quote anything we covered in class in support of your reasoning.

Problem 1

Let $f(x) = x^2$. Compute the upper and lower Riemann sums for the intervals $[0, 1]$ and the partition P_n with $x_0 = 0$, $x_j = \frac{j}{n}$, $x_n = 1$, and show directly that f is Riemann integrable by evaluating the limits of the upper and lower sums. You may use the identity

$$\sum_{j=1}^{n} j^2 = \frac{n(n+1)(2n+1)}{6}\,.$$

Problem 2

Let

$$f(x) = \text{sgn}(\sin(\frac{\pi}{x})) \text{ if } x \neq 0 \text{ and } f(0) = 0$$

where

$$\text{sgn}(x) = 1 \text{ if } x > 0, \quad \text{sgn}(x) = -1 \text{ if } x < 0 \text{ and } \text{sgn}(0) = 0\,.$$

Show that f is Riemann integrable on the interval $[0, 1]$.

Problem 3

Let $\alpha : [a, b] \to \mathbb{R}$ be an increasing function and let $f : [a, b] \to \mathbb{R}$ be a bounded function. If $m = \inf\{f(x) : a \leq x \leq b\}$ and $M = \sup\{f(x) : a \leq x \leq b\}$, then prove that $m(\alpha(b) - \alpha(a)) \leq \underline{\int_a^b} f d\alpha \leq \overline{\int_a^b} f d\alpha \leq M(\alpha(b) - \alpha(a))$. You may quote results from class to support your argument.

Problem 4

Let $\alpha : [-2, 2] \to \mathbb{R}$ be the function associated with the coin flip, $\alpha(x) = 0$ if $x < -1$, $\alpha(x) = p$ if $-1 \leq x < 1$ and $\alpha(x) = 1$ if $x \geq 1$. Compute $\mu = \int_{-2}^{2} x d\alpha(x)$ and $\sigma^2 = \int_{-2}^{2} (x - \mu)^2 d\alpha(x)$.