5.2. COROLLARY. *Let (X, d) and (Y, ρ) be metric spaces, and let $f : X \to Y$ be a continuous function. If X is compact, then $f(X)$ is a bounded subset of Y.*

Thus, when X is non-empty, compact and $f : X \to \mathbb{R}$ is continuous, then there exists M such that $|f(x)| \leq M$ for every $x \in X$. So in particular $\sup\{f(x) : x \in X\}$ and $\inf\{f(x) : x \in X\}$ exists. The following shows that not only does the supremum and infimum exist but that they are attained.

5.3. COROLLARY. *Let (X, d) be a non-empty compact metric space and let $f : X \to \mathbb{R}$ be continuous. Then there are points $x_m, x_M \in X$, such that for any $x \in X$, $f(x_m) \leq f(x) \leq f(x_M)$. That is, $f(x_m) = \inf\{f(x) : x \in X\}$ and $f(x_M) = \sup\{f(x) : x \in X\}$.*

The above result gives the proof that whenever $f : [a, b] \to \mathbb{R}$ is continuous, then f attains its maximum and minimum value. First, by Heine-Borel the interval $[a, b]$ is compact, now apply the above result. Note that $(0, 1)$ is a bounded interval, $f(x) = 1/x$ is continuous on this set but is not even bounded.

5.4. THEOREM. *Let (X, d) and (Y, ρ) be metric spaces, and let $f : X \to Y$ be a continuous function. If X is compact, then f is uniformly continuous.*

Thus, for example any rational function $r(x) = p(x)/q(x)$, such that $q(x) \neq 0$ on the set $[a, b]$ will be uniformly continuous.

2.6. Connected Sets and the Intermediate Value Theorem

We will see in this section that the intermediate value theorem from calculus is really a consequence of the fact that an interval of real numbers is a connected set. First we need to define this concept.

6.1. DEFINITION. *A metric space (X, d) is* **connected** *if the only subsets of X that are both open and closed are X and the empty set. A subset S of X is called* **connected** *provided that the subspace (S, d) is a connected metric space. If S is not connected then we say that S is* **disconnected** *or* **separated.**

6.2. PROPOSITION. *A metric space (X, d) is disconnected if and only if X can be written as a union of two disjoint, non-empty open sets.*

6.3. EXAMPLE. *If (X, d) is a discrete metric space with two or more points, then X is disconnected since $X = \{p_0\} \cup \{p_0\}^c$ expresses X as a disjoint union of two non-empty open sets.*

We now come to perhaps the most important example of a connected space. By an *interval* in $\mathbb{R}$ we mean either an open interval, closed interval, or half-open interval. The endpoints can be either an actual number or $+\infty$ or $-\infty$.

6.4. THEOREM. *Let $I \subseteq \mathbb{R}$ be an interval or all of $\mathbb{R}$. Then I is a connected set.*

Now we come to a general version of the Intermediate Value Theorem.

6.5. THEOREM (Intermediate Value Theorem for Metric Spaces). *Let* (X, d) *be a connected metric space and let* $f : X \to \mathbb{R}$ *be a continuous function. If* $x_0, x_1 \in X$ *with* $f(x_0) < L < f(x_1)$, *then there is* $x_2 \in X$ *with* $f(x_2) = L$.

6.6. COROLLARY (Intermediate Value Theorem). *Let* $I \subseteq \mathbb{R}$ *be an interval or the whole real line and let* $f : I \to \mathbb{R}$ *be continuous. If* $x_0, x_1 \in I$ *and* $f(x_0) < L < f(x_1)$, *then there is* x_2 *between* x_0 *and* x_1 *with* $f(x_2) = L$.

6.7. THEOREM. *Let* (X, d) *and* (Y, ρ) *be metric spaces and let* $f : X \to Y$ *be continuous. If* X *is connected, then* $f(X) \subseteq Y$ *is a connected subset.*

6.8. DEFINITION. *A metric space (X, d) is called* **pathwise connected** *provided that for any two points, $a, b \in X$ there exists a continuous function $f : [0, 1] \to X$ such that $f(0) = a$ and $f(1) = b$.*

Intuitively, a space is pathwise connected if and only if you can draw a "curve" between any two points with no breaks in the curve.

6.9. EXAMPLE. *The subset of the plane defined by*

$$X = \{(x, sin(\frac{1}{x})) : 0 < x \leq 1\} \cup \{(0, y) : -1 \leq y \leq +1\}$$

is a space that is connected, but not pathwise connected.

CHAPTER 3

The Contraction Mapping Principle

At this point we would like to present an important result that uses the concepts that we have developed and together with some of its applications. Logically, this material should be done later, since it uses facts from calculus about derivatives and integrals which we have not yet discussed. But we find that a little practical math at this stage helps to motivate the rest of the course.

3.1. Contraction Mappings

1.1. DEFINITION. *Let (X, d) be a metric space. A function $f : X \to X$ is called a* **contraction mapping** *provided that there is $r, 0 < r < 1$, so that $d(f(x), f(y)) \leq rd(x, y)$ for every $x, y \in X$.*

Note that saying that f is a contraction mapping is the same as saying that it is Lipschitz continuous with constant $r < 1$. It is important to note that when we say that f is a contraction mapping that is stronger than just saying that $d(f(x), f(y)) < d(x, y)$, since we need the r.
Given a function $f : X \to X$ any point satisfying $f(x_0) = x_0$ is called a **fixed point** of the function.

1.2. THEOREM (Contraction Mapping Principle). *Let (X, d) be a complete metric space and let $f : X \to X$ be a contraction mapping. Then:*

(1) *there exists a unique point $x_0 \in X$, such that $f(x_0) = x_0$,*
(2) *if $x_1 \in X$ is any point and we define a sequence inductively, by setting $x_{n+1} = f(x_n)$, then $\lim_n x_n = x_o$,*
(3) *for this sequence, we have that $d(x_0, x_n) \leq \frac{d(x_2, x_1)r^{n-1}}{1-r}$.*

Thus, the contraction mapping principle not only guarantees us the existence of a fixed point, but shows that there is a unique fixed point and gives us a method for approximating the fixed point, together with an estimate of how close the sequence x_n is to the fixed point! This is a remarkable amount of information.

Here's a typical application of this theorem. Let $f(x) = cos(x)$ and note that since $1 < \pi/2$ for $0 \leq x \leq 1$, we have that $0 \leq f(x) \leq 1$. Thus, $f : [0,1] \to [0,1]$ and is continuous. Also, by the mean value theorem, for $0 \leq x \leq y \leq 1$, there is $c, x \leq c \leq y$ with

$$f(y) - f(x) = f'(c)(y - x) = -sin(c)(y - x).$$

Hence, $|f(y) - f(x)| \leq sin(c)|y - x| \leq sin(1)|y - x|$. Thus, $f(x) = cos(x)$ is a contraction mapping with $r = sin(1) < 1$.

Using the fact that $[0,1]$ is complete, by the contraction mapping principle, there is a unique point, $0 \leq x_0 \leq 1$, such that $cos(x_0) = x_0$. Moreover, we can obtain this point(or at least approximate it) by choosing any number $x_1, 0 \leq x_1 \leq 1$ and forming the inductive sequence $x_{n+1} = cos(x_n)$.

The third part of the theorem gives us an estimate of the distance between our "approximate" fixed point x_n and the true fixed point. In particular, if we pick $x_1 = 0$, then $x_2 = cos(x_1) = 1$, so $|x_2 - x_1| = 1$. Hence, $|x_0 - x_n| \leq \frac{sin(1)^{n-1}}{1-sin(1)}$.

3.2. Application: Newton's Method

Newton's method gives an iterative method for approximating the solution to an equation of the form $f(x) = 0$, when f is differentiable.

The idea of the iteration is to choose any x_1 and then get an "improved" estimate to the solution by tracing the tangent line to the graph at the point $(x_1, f(x_1))$ until it intersects the x-axis and letting this determine the point x_2. The formula that one obtains is

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}.$$

Newton's method consists of repeating this formula iteratively to generate a sequence of points

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

that, hopefully, converge to the zero of the function.

Note that if we set $g(x) = x - \frac{f(x)}{f'(x)}$, then $f(x) = 0$ if and only if $g(x) = x$. Thus, finding a zero of f is equivalent to finding a fixed point of g. Moreover, the above iteration is simply computing $x_{n+1} = g(x_n)$.

Thus, if we can construct an interval $[a, b]$ such that $g : [a, b] \to [a, b]$ and such that g is a contraction mapping on $[a, b]$ then we will have a criterion for convergence of Newton's method.

The details are below.

2.1. THEOREM (Newton's Method). *Let f be a twice continuously differentiable function and assume that there is a point x_0 with $f(x_0) = 0$ and $f'(x_0) \neq 0$. Then there is $M > 0$ so that for any x_1 with $x_0 - M \leq x_1 \leq x_0 + M$ the sequence of points obtained by Newton's method starting at x_1 converges to x_0.*