Math 4331
Introduction to Real Analysis

A summary of notes by Vern I. Paulsen
edited by Bernhard G. Bodmann

Contents

CHAPTER 1

Metric Spaces

This is a summary of the material covered in the *Introduction to Real Analysis* course given at the University of Houston.

1.1. Definition and Examples

1.1. DEFINITION. *Given a set X a* **metric** *on X is a function $d : X \times X \to \mathbb{R}$ satisfying:*

(1) *for every $x, y \in X$, $d(x, y) \geq 0$,*
(2) *$d(x, y) = 0$ if and only if $x = y$,*
(3) *$d(x, y) = d(y, x)$,*
(4) *(triangle inequality) for every $x, y, z \in X$, $d(x, y) \leq d(x, z) + d(z, y)$.*

The pair (X, d) is called a **metric space.**

1.2. EXAMPLE. *Let $X = \mathbb{R}$ and set $d(x, y) = |x - y|$, then d is a metric on $\mathbb{R}$. We call this the* usual metric *on $\mathbb{R}$.*

To prove it is a metric we verify (1)–(4). For (1): $d(x, y) = |x - y| \geq 0$, by the definition of the absolute value functions so (1). Since $d(x, y) = 0$ if and only if $|x - y| = 0$ if and only if $x = y$, (2) follows. (3) follows since $d(x, y) = |x - y| = |y - x| = d(y, x)$. Finally, for (4), $d(x, y) = |x - y| = |x - z + z - y| \leq |x - z| + |z - y| = d(x, z) + d(z, y)$.

1.3. EXAMPLE (The taxi cab metric). *Let $X = \mathbb{R}^2$. Given $x = (x_1, x_2), y = (y_1, y_2)$, set $d(x, y) = |x_1 - y_1| + |x_2 - y_2|$, then d is a metric on $\mathbb{R}^2$.*

We verify (1)–(4). (1) and (3) are obvious. For (2): $d(x, y) = 0$ iff $|x_1 - y_1| + |x_2 - y_2| = 0$. But since both terms in the sum are non-negative for the sum to be 0, each one must be 0. So $d(x, y) = 0$ iff $|x_1 - y_1| = 0$ AND $|x_2 - y_2| = 0$ iff $x_1 = y_1$ and $x_2 = y_2$ iff $x = (x_1, x_2) = (y_1, y_2) = y$. Finally to see (4):

$$\begin{aligned} d(x, y) &= |x_1 - y_1| + |x_2 - y_2| = |x_1 - z_1 + z_1 - y_1| + |x_2 - z_2 + z_2 - y_2| \\ &\leq |x_1 - z_1| + |z_1 - y_1| + |x_2 - z_2| + |z_2 - y_2| = d(x, z) + d(z, y). \end{aligned}$$

We often denote the taxi cab metric by $d_1(x, y)$.

1.4. EXAMPLE. *A different metric on $\mathbb{R}^2$. For $x = (x_1, x_2), y = (y_1, y_2)$ set $d_\infty(x, y) = \max\{|x_1 - y_1|, |x_2 - y_2|\}$. So the distance between two points is the larger of these two numbers.*

We only check the triangle inequality. Let $z = (z_1, z_2)$ be another point. We have two cases to check. Either $|x_1 - y_1| = d(x, y)$ OR $d(x, y) = |x_2 - y_2|$.
Case 1: $d(x, y) = |x_1 - y_1|$.
Now notice that $|x_1 - z_1| \leq \max\{|x_1 - z_1|, |x_2 - z_2|\} = d(x, z)$. Similarly, $|z_1 - y_1| \leq \max\{|z_1 - y_1|, |z_2 - y_2|\}$. Hence,

$$\begin{aligned} d(x, y) = |x_1 - y_1| = |x_1 - z_1 + z_1 - y_1| \leq |x_1 - z_1| + |z_1 - y_1| \\ \leq d(x, z) + d(z, y). \end{aligned}$$

Case 2: $d(x, y) = |x_2 - y_2|$. Now use $|x_2 - z_2| \leq \max\{|x_1 - z_1|, |x_2 - z_2|\} = d(x, z)$. Similarly, $|z_2 - y_2| \leq \max\{|z_1 - y_1|, |z_2 - y_2|\}$. Hence,

$$\begin{aligned} d(x, y) = |x_2 - y_2| = |x_2 - z_2 + z_2 - y_2| \leq |x_2 - z_2| + |z_2 - y_2| \\ \leq d(x, z) + d(z, y). \end{aligned}$$

So in each case the triangle inequality is true, so it is true.

1.5. EXAMPLE. *In this case we let X be the set of all continuous real-valued functions on $[0, 1]$. We use three facts from Math 3333:*

(1) *if f and g are continuous on [0,1], then $f - g$ is continuous on [0,1],*
(2) *if f is continuous on [0,1], then $|f|$ is continuous on [0,1],*
(3) *if h is continuous on [0,1], then there is a point $0 \leq t_0 \leq 1$, so that $h(t) \leq h(t_0)$ for every $0 \leq t \leq 1$. That is $h(t_0) = \max\{h(t) : 0 \leq t \leq 1\}$.*

Now given $f, g \in X$, we set $d(f, g) = \max\{|f(t) - g(t)| : 0 \leq t \leq 1\}$.
Note that by (1) and (2) $|f - g|$ is continuous and so by (3) there is a point where it acheives its maximum.

We now show that d is a metric on X. Clearly, (1) holds. Next, if $d(f, g) = 0$, then the maximum of $|f(t) - g(t)|$ is 0, so we must have that $|f(t) - g(t)| = 0$ for every t. But then this means that $f(t) = g(t)$ for every t, and so $f = g$. So $d(f, g) = 0$ implies $f = g$. Also $f = g$ implies $d(f, g) = 0$ so (2) holds. Clearly (3) holds. Finally to see the triangle inequality, we let f, g, h be three continuous functions on [0,1]. that is, $f, g, h \in X$. We must show that $d(f, g) \leq d(f, h) + d(h, g)$.
We know that there is a point $t_0, 0 \leq t_0 \leq 1$, so that

$$d(f, g) = \max\{|f(t) - g(t)| : 0 \leq t \leq 1\} = |f(t_0) - g(t_0)|.$$

Hence,

$$\begin{aligned} d(f, g) = |f(t_0) - g(t_0)| &= |f(t_0) - h(t_0) + h(t_0) - g(t_0)| \\ &\leq |f(t_0) - h(t_0)| + |h(t_0) - g(t_0)| \\ &\leq \max\{|f(t) - h(t)| : 0 \leq t \leq 1\} + \max\{|h(t) - g(t)| : 0 \leq t \leq 1\} \\ &= d(f, h) + d(h, g) \end{aligned}$$

1.6. EXAMPLE (Euclidean space, Euclidean metric). *Let $X = \mathbb{R}^n$ the set of real n-tuples. For $x = (a_1, \ldots, a_n)$ and $y = (b_1, \ldots, b_n)$ we set*

$$d(x,y) = \sqrt{(a_1 - b_1)^2 + \cdots + (a_n - b_n)^2}.$$

This efines a metric on $\mathbb{R}^n$, which we will prove shortly. This metric is called the **Euclidean metric** *and $(\mathbb{R}^n, d)$ is called* **Euclidean space.**

It is easy to see that the Euclidean metric satisfies (1)–(3) of a metric. It is harder to prove the triangle inequality for the Euclidean metric than some of the others that we have looked at. This requires some results first.

1.7. LEMMA. *Let $p(t) = at^2 + bt + c$ with $a \geq 0$. If $p(t) \geq 0$ for every $t \in \mathbb{R}$, then $b^2 \leq 4ac$.*

1.8. PROPOSITION (Schwarz Inequality). *Let $a_1, ..., a_n, b_1, ..., b_n$ be real numbers. Then*

$$|a_1 b_1 + \cdots a_n b_n| \leq \sqrt{a_1^2 + \cdots + a_n^2}\sqrt{b_1^2 + \cdots + b_n^2}.$$

1.9. COROLLARY. $\sqrt{(a_1 + b_1)^2 + \cdots + (a_n + b_n)^2} \leq \sqrt{a_1^2 + \cdots + a_n^2} + \sqrt{b_1^2 + \cdots + b_n^2}.$

Now prove the triangle inequality.

1.10. EXAMPLE (The discrete metric). *Let X be any non-empty set and define*

$$d(x,y) = \begin{cases} 1 & x \neq y \\ 0 & x = y \end{cases}.$$

Then this is a metric on X called the **discrete metric** *and we call (X, d) a* **discrete metric space.**

1.2. Open Sets

2.1. DEFINITION. *Let (X,d) be a metric space, fix $x \in X$ and $r > 0$. The* **open ball of radius r centered at x** *is the set*

$$B(x;r) = \{y \in X : d(x,y) < r\}.$$

2.2. EXAMPLE. *In $\mathbb{R}$ with the usual metric, $B(x;r) = \{y : |x - y| < r\} = \{y : x - r < y < x + r\} = (x - r, x + r)$.*

2.3. EXAMPLE. *In $\mathbb{R}^2$ with the Euclidean metric, $x = (x_1, x_2)$, then $B(x;r) = \{(y_1, y_2) : (x_1 - y_1)^2 + (x_2 - y_2)^2 < r^2\}$, which is a disk of radius r centered at x.*

2.4. EXAMPLE. *In $\mathbb{R}^3$ with the Euclidean metric, $B(x;r)$ really is an open ball of radius r. This example is where the name comes from.*

2.5. EXAMPLE. *In $(\mathbb{R}^2, d_\infty)$ we have*

$$B(x;r) = \{(y_1, y_2) : |x_1 - y_1| < r \text{ and } |x_2 - y_2| < r\} = \\ \{(y_1, y_2) : x_1 - r < y_1 < x_1 + r \text{ and } x_2 - r < y_2 < y_2 + r\}.$$

So now an open "ball" is actually an open square, centered at x with sides of length 2r.

2.6. EXAMPLE. *In $(\mathbb{R}^2, d_1)$ we have $B((0,0);1) = \{(x,y) : |x-0|+|y-0| < 1\}$ which can be seen to be the "diamond" with corners at (1,0),(0,1),(-1,0), (0,-1).*

2.7. EXAMPLE. *When $X = \{f : [0,1] \to \mathbb{R} | f$ is continuous$\}$ and $d(f,g) = \max\{|f(t)-g(t)| : 0 \leq t \leq 1\}$, then $B(f;r) = \{g : g$ is continuous and $f(t)-r < g(t) < f(t)+r, \forall t\}$. This can be pictured as all continuous functions g whose graphs lie in a band of width r about the graph of f.*

2.8. EXAMPLE. *If we let $\mathbb{R}$ have the usual metric and let $Y = [0,1] \subseteq \mathbb{R}$ be the subspace, then when we look at the metric space Y we have that $B(0;1/2) = [0,1/2) = (-1/2,1/2) \cap Y$.*

2.9. EXAMPLE. *When we let X be a set with the discrete metric and $x \in X$, then $B(x;r) = \{x\}$ when $r < 1$. When $r \geq 1$, then $B(x;r) = X$.*

2.10. DEFINITION. *Given a metric space (X,d) a subset $\mathcal{O} \subseteq X$ is called* **open** *provided that whenever $x \in \mathcal{O}$, then there is an $r > 0$ such that $B(x;r) \subseteq \mathcal{O}$.*

Showing that sets are open really requires proof, so we do a few examples.

2.11. EXAMPLE. *In $\mathbb{R}$ with the usual metric, an interval of the form $(a,b) = \{x : a < x < b\}$ is an open set.*

So "open intervals" really are "open sets".

2.12. EXAMPLE. *In $\mathbb{R}^2$ with the Euclidean metric, a rectangle of the form $R = \{(y_1,y_2) : a < y_1 < b, c < y_2 < d\}$ is an open set.*

2.13. EXAMPLE. *In $\mathbb{R}$ with the usual metric an interval of the form $[a,b)$ is not open.*

CAREFUL! If we let $Y = [0,1) \subseteq \mathbb{R}$ be equipped with the metric from $\mathbb{R}$. Then *in* Y the set $\mathcal{O} = [0,1/2)$ is open! (Explain why.)
The next result justifies us calling $B(x;r)$ an *open* ball.

2.14. PROPOSITION. *Let (X,d) be a metric space, fix $x \in X$ and $r > 0$. Then $B(x;r)$ is an open set.*

2.15. THEOREM. *Let (X,d) be a metric space. Then*

(1) *the empty set is open,*
(2) *X is open,*
(3) *the union of any collection of open sets is open,*
(4) *the intersection of finitely many open sets is open.*

2.16. PROPOSITION. *In a discrete metric space, every set is open.*

Uniformly Equivalent Metrics. The definition of open set really depends on the metric. For example, on $\mathbb{R}$ if instead of the usual metric we used the discrete metric, then every set would be open. But we have seen that when $\mathbb{R}$ has the usual metric, then not every set is open. For example, $[a, b)$ is not an open subset of $\mathbb{R}$ in the usual metric. Thus, whether a set is open or not really can depend on the metric that we are using.
For this reason, if a given set X has two metrics, d and ρ, and we say that a set is open, we generally need to specify which metric we mean. Consequently, we will say that a set is **open with respect to** d or **open in (X, d)** when we want to specify that it is open when we use the metric d. In this case it may or may not be **open with respect to** ρ.
In the case of $\mathbb{R}^2$, we already have three metrics, the Euclidean metric d, the taxi cab metric d_1 and the metric d_∞. So when we say that a set is open in $\mathbb{R}^2$, we could potentially mean three different things. On the other hand it could be the case that all three of these metrics give rise to the same collection of open sets.
In fact, these three metrics do give rise to the same collections of open sets and the following definition and result explains why.

2.17. DEFINITION. *Let X be a set and let d and ρ be two metrics on X. We say that these metrics are* **uniformly equivalent** *provided that there are constants A and B such that for every $x, y \in X$,*

$$\rho(x, y) \leq A d(x, y) \text{ and } d(x, y) \leq B\rho(x, y).$$

2.18. EXAMPLE. *On $\mathbb{R}^2$ the Euclidean d and the metric d_∞ are uniformly equivalent. In fact,*

$$d_\infty(x, y) \leq d(x, y) \text{ and } d(x, y) \leq \sqrt{2} d_\infty(x, y).$$

To see this, let $x = (a_1, a_2), y = (b_1, b_2)$. Since $|a_1 - b_1| \leq \sqrt{(a_1 - b_1)^2 + (a_2 - b_2)^2}$ and $|a_2 - b_2| \leq \sqrt{(a_1 - b_1)^2 + (a_2 - b_2)^2}$, we have that

$$d_\infty(x, y) = \max\{|a_1 - b_1|, |a_2 - b_2|\} \leq \sqrt{(a_1 - b_1)^2 + (a_2 - b_2)^2} = d(x, y).$$

On the other hand, since $|a_1 - b_1| \leq d_\infty(x, y)$ and $|a_2 - b_2| \leq d_\infty(x, y)$, we have that $(a_1 - b_1)^2 + (a_2 - b_2)^2 \leq 2(d_\infty(x, y))^2$. Taking square roots of both sides, yields $d(x, y) \leq \sqrt{2} d_\infty(x, y)$.

2.19. EXAMPLE. *On $\mathbb{R}^2$ the Euclidean metric d and d_1 are uniformly equivalent. In fact,*

$$d(x, y) \leq d_1(x, y) \text{ and } d_1(x, y) \leq \sqrt{2} d(x, y).$$

We have that

$$\begin{aligned} d_1(x, y)^2 &= (|a_1 - b_1| + |a_2 - b_2|)^2 \\ &= |a_1 - b_1|^2 + 2|a_1 - b_1||a_2 - b_2| + |a_2 - b_2|^2 \geq (a_1 - b_1)^2 + (a_2 - b_2)^2 \\ &= (d(x, y))^2. \end{aligned}$$

Hence, $d(x,y) \leq d_1(x,y)$.
To see the other inequality, we use the Schwarz inequality,

$$d_1(x,y) = ||a_1 - b_1| \cdot 1 + |a_2 - b_2| \cdot 1| \leq \sqrt{|a_1 - b_1|^2 + |a_2 - b_2|^2}\sqrt{1^2 + 1^2} = \sqrt{2}d(x,y)$$

Given a set X with metrics d and ρ, a point $x \in X$ and $r > 0$, we shall write $B_d(x;r) = \{y \in X : d(x,y) < r\}$ and $B_\rho(x;r) = \{y \in X : \rho(x,y) < r\}$.

2.20. LEMMA. *Let X be a set, let d and ρ be two metrics on X that are uniformly equivalent, and let A and B denote the constants that appear in Definition 2.17. Then $B_\rho(x;r) \subseteq B_d(x;Br)$ and $B_d(x;r) \subseteq B_\rho(x;Ar)$.*

2.21. THEOREM. *Let X be a set and let d and ρ be metrics on X that are uniformly equivalent. Then a set is open with respect to d if and only if it is open with respect to ρ.*

1.3. Closed Sets

3.1. DEFINITION. *Given a set X and $E \subseteq X$, the* **complement of E,** *denoted E^c is the set of all elements of X that are not in E, i.e.,*
$E^c = \{x \in X : x \notin E\}$.

Other notations that are used for the complement are $E^c = \mathcal{C}E = X \backslash E$.
Note that $(E^c)^c = E$.

3.2. DEFINITION. *Let (X,d) be a metric space. Then a set $E \subseteq X$ is closed if and only if E^c is open.*

The following gives a useful way to re-state this definition.

3.3. PROPOSITION. *Let (X,d) be a metric space. Then a set E is closed if and only if there is an open set $\mathcal{O}$ such that $E = \mathcal{O}^c$.*

3.4. EXAMPLE. *In $\mathbb{R}$ with the usual metric, we have that (b,∞) is open and $(-\infty,a)$ is open. So when $a < b$ we have that $\mathcal{O} = (-\infty,a) \cup (b,+\infty)$ is open. Hence, $\mathcal{O}^c = [a,b]$ is closed.*

This shows that our old calculus definition of a "closed interval", really is a closed set in this sense.

3.5. DEFINITION. *Let (X,d) be a metric space, let $x \in X$ and let $r > 0$. The* **closed ball with center x and radius r** *is the set*

$$B^-(x;r) = \{y \in X : d(x,y) \leq r\}.$$

The following result explains this notation.

3.6. PROPOSITION. *Let (X, d) be a metric space, let $x \in X$ and let $r > 0$. Then $B^-(x;r)$ is a closed set.*

Because the definition of closed sets involves complements, it is useful to recall DeMorgan's Laws. Given subsets $E_i \subseteq X$ where i belongs to some set I, we have

$$\bigcup_{i \in I} E_i = \{x \in X : \text{ there exists } i \in I \text{ with } x \in E_i\}$$

and

$$\bigcap_{i \in I} E_i = \{x \in X : x \in E_i \text{ for every } i \in I\}.$$

3.7. PROPOSITION (DeMorgan). *Let $E_i \subseteq X$ for $i \in I$. Then*

$$(\bigcup_{i \in I} E_i)^c = \bigcap_{i \in I} E_i^c \text{ and } (\bigcap_{i \in I} E_i)^c = \bigcup_{i \in I} E_i^c.$$

The following theorem about closed sets follows from DeMorgan's Laws and Theorem 1.31.

3.8. THEOREM. *Let (X, d) be a metric space. Then:*

1. *the empty set is closed,*
2. *X is closed,*
3. *the intersection of any collection of closed sets is closed,*
4. *the union of finitely many closed sets is closed.*

3.9. EXAMPLE. *Consider $\mathbb{R}$ equipped with the usual metric. Let $C_1 = [0, 1]$, $C_2 = C_1 \setminus (1/3, 2/3)$, $C_3 = C_2 \setminus ((1/9, 2/9) \cup (4/9, 5/9) \cup (7/9, 8/9))$ and proceed accordingly. Then, each C_j is closed and thus $C = \cap_{j=1}^{\infty} C_j$ is closed as well. This set is known as the Cantor set.*

3.10. PROPOSITION. *In a discrete metric space, every set is closed.*

As with open sets, when there is more than one metric on the set X, then we need to specify which metric we are referring to when saying that a set is closed. The following is the analogue of Theorem 1.41 for closed sets.

3.11. PROPOSITION. *Let X be a set and let d and ρ be two metrics on X that are uniformly equivalent. Then a set is closed with respect to d if and only if it is closed with respect to ρ.*

1.4. Convergent Sequences

Our general idea from calculus of a sequence $\{p_n\}$ converging to a point p is that as n grows larger, the points p_n grow closer and closer to p. When we say "grow closer" we really have in our minds that some distance is growing smaller. This leads naturally to the following definition.

4.1. DEFINITION. *Let (X, d) be a metric space, $\{p_n\} \subseteq X$ a sequence in X and $p \in X$. We say that the sequence $\{p_n\}$* **converges to p** *and write*

$$\lim_{n \to +\infty} p_n = p,$$

provided that for every $\epsilon >$), there is a real number N so that when $n > N$, then $d(p, p_n) < \epsilon$.

Often out of sheer laziness, I will write $\lim_n p_n = p$ for $\lim_{n\to+\infty} p_n = p$.

4.2. EXAMPLE. *When $X = \mathbb{R}$ and d is the usual metric, then $d(p, p_n) = |p - p_n|$ and this definition is identical to the definition used when we studied convergent sequences of real numbers in Math 3333.*

For a quick review, we will look at a couple of examples of sequences and recall how we would prove that they converge.
For a first example, consder the sequence given by $p_n = \frac{3n+1}{5n-2}$. Fo any natrual number n, the denominator of this fraction is non-zero. So this formula defines a sequence of points. For large n, the $+1$ in the numerator and the -2 in the denominator are small in relation to the $3n$ and $5n$ so we expect that this sequence has limit $p = \frac{3}{5}$.
Now for some scrap work. To prove this we would need

$$d(p, p_n) = |p - p_n| = |\frac{3(5n-2) - (3n+1)5}{5(5n-2)}| < \epsilon.$$

Simplifying this fraction leads to the condition $\frac{1}{25n-10} < \epsilon$. Solving for n we see that this is true provided $\frac{1}{\epsilon} < 25n - 10$, i.e., $\frac{1}{25\epsilon} + \frac{2}{5} < n$. So it looks like we should choose, $N = \frac{1}{25\epsilon} + \frac{2}{5}$. This is not a proof, just our scrap work.
Now for the proof:
Given $\epsilon > 0$, define $N = \frac{1}{25\epsilon} + \frac{2}{5}$. For any $n > N$, we have that $25n - 10 > 25N - 10 = \frac{1}{\epsilon}$, and so $d(p, p_n) = |\frac{3}{5} - \frac{3n+1}{5n-2}| = \frac{1}{25n-10} < (\frac{1}{\epsilon})^{-1} = \epsilon$. Hence, $\lim_{n\to+\infty} p_n = p$.
For the next example, we look at $p_n = \sqrt{n^2 + 8n} - n$ and prove that this sequence has limit $p = 4$.
Now that we have recalled what this definition means in $\mathbb{R}$ and have seen how to prove a few examples. We want to look at what this concept means in some of our other favorite metric spaces.

4.3. THEOREM. *Let $\mathbb{R}^k$ be endowed with the Euclidean metric d, let $\{p_n\} \subseteq \mathbb{R}^k$ be a sequence with $p_n = (a_{1,n}, a_{2,n}, \ldots, a_{k,n})$ and let $p = (a_1, a_2, \ldots, a_k) \in \mathbb{R}^k$. Then $\lim_{n\to+\infty} p_n = p$ in $(\mathbb{R}^k, d)$ if and only if for each $j, 1 \le j \le k$, we have $\lim_{n\to+\infty} a_{j,n} = a_j$ in $\mathbb{R}$ with the usual metric.*

So the crux of the above theorem is that a sequence of points in $\mathbb{R}^k$ converges if and only if each of their components converge.
If we combine our first two examples with the above theorem, we see that if we define a sequence of points in $\mathbb{R}^2$ by setting $p_n = (\frac{3n+1}{5n-2}, \sqrt{n^2 + 8n} - n)$ then in the Euclidean metric these points converge to the point $p = (\frac{3}{5}, 4)$.
What if we had used the taxi cab metric or d_∞ metric on $\mathbb{R}^2$ instead of the Euclidean metric, would these points still converge to the same point? The answer is yes and the following result explains why and saves us having to prove separate theorems for each of these metrics.

4.4. PROPOSITION. *Let X be a set with metrics d and ρ that are uniformly equivalent, let $\{p_n\} \subseteq X$ be a sequence and let $p \in X$ be a point. Then*

$\{p_n\}$ converges to p in the metric d if and only if $\{p_n\}$ converges to p in the metric ρ.

We now look at the discrete metric.

4.5. PROPOSITION. *Let (X, d) be the discrete metric space, let $\{p_n\} \subseteq X$ be a sequence, and let $p \in X$. Then the sequence $\{p_n\}$ converges to p if and only if there exists N so that for $n > N$, $p_n = p$.*

A sequence $\{p_n\}$ such that $p_n = p$ for every $n > N$, is often called **eventually constant.**
We now look at some general theorems about convergence in metric spaces.

4.6. PROPOSITION. *Let (X, d) be a metric space. A sequence $\{p_n\} \subseteq X$ can have at most one limit.*

The above result is most often used to prove that two points are really the same point, since if $\lim_n p_n = p$ and $\lim_n p_n = q$, then $p = q$.
Convergence is an important way to characterize closed sets.

4.7. THEOREM. *Let (X, d) be a metric space and let $S \subseteq X$ be a subset. Then S is a closed subset if and only if whenever $\{p_n\} \subseteq S$ is a convergent sequence, we have $\lim_n p_n \in S$.*

That is, a set is closed if and only if limits of convergent sequences stay in the set.
One of our other main results from Math 3333 about convergent seqeunces in $\mathbb{R}$ is that every convergent sequence of real numbers is bounded. This plays an important role in metric spaces too. But first we need to say what it means for a set to be bounded in a metric space.

4.8. PROPOSITION. *Let (X, d) be a metric space and let $E \subseteq X$ be a subset. Then the following are equivalent:*

(1) *there exists a point $p \in X$ and $r_1 > 0$ such that $E \subseteq B^-(p; r_1)$,*
(2) *there exists a point $q \in X$ and $r_2 > 0$ such that $E \subseteq B(q; r_2)$,*
(3) *there exists $M > 0$ so that every $x, y \in E$ satisfies $d(x, y) \leq M$.*

4.9. DEFINITION. *Let (X, d) be a metric space and let $E \subseteq X$. We say that E is a* **bounded set** *provided that it satisfies any of the three equivalent conditions of the above proposition.*

4.10. PROPOSITION. *Let (X,d) be a metric space. If $\{p_n\}$ is a convergent sequence, then it is bounded.*

Recall that $\mathbb{R}^k$ is a vector space, for $p = (a_1, \ldots, a_k), q = (b_1, \ldots, b_k) \in \mathbb{R}^k$ and $r \in \mathbb{R}$ we have

$$p + q = (a_1 + b_1, \ldots, a_k + b_k) \text{ and } rp = (ra_1, \ldots, ra_k).$$

There is also the "dot product",

$$p \cdot q = a_1 b_1 + \cdots + a_k b_k.$$

Note that in terms of the dot product and Euclidean metric, we can see that the Schwarz inequality says that

$$|p \cdot q| \leq \sqrt{a_1^2 + \cdots + a_k^2}\sqrt{b_1^2 + \cdots + b_k^2} = d(0,p)d(0,q).$$

Two other useful connection to notice between the Euclidean metric and the vector space operations is that

$$d(p,q) = \sqrt{(a_1 - b_1)^2 + \cdots + (a_k - b_k)^2} = d(0, p - q)$$

and

$$d(rp, rq) = \sqrt{(ra_1 - rb_1)^2 + \cdots + (ra_k - rb_k)^2} = |r|d(p,q).$$

4.11. LEMMA. *Let $(\mathbb{R}^k, d)$ be Euclidean space. If $E \subseteq \mathbb{R}^k$ is a bounded set, then there is A so that $E \subseteq B^-(0; A)$.*

The following result can be proved using Theorem 1.57 and results from Math 3333 about convergence of sums and product of real numbers. we give a proof that mimics the proofs for real numbers.

4.12. THEOREM. *Let $(\mathbb{R}^k, d)$ be Euclidean space, let $p_n = (a_{1,n}, \ldots, a_{k,n})$ and $q_n = (b_{1,n}, \ldots, b_{k,n})$ be sequences in $\mathbb{R}^k$ with $\lim_n p_n = p = (a_1, \ldots, a_k)$ and $\lim_n q_n = q = (b_1, \ldots, b_k)$, and let $\{r_n\}$ be a sequence in $\mathbb{R}$ with $\lim_n r_n = r$. Then we have the following:*

(1) $\lim_n p_n + q_n = p + q$,
(2) $\lim_n r_n p_n = rp$,
(3) $\lim_n p_n \cdot q_n = p \cdot q$.

1.5. Interiors, Closures, Boundaries of Sets

In this section we look at some other important concepts related to open and closed sets.

5.1. DEFINITION. *Let (X,d) be a metric space and let $A \subseteq X$. Then the* **closure of A,** *denoted $\overline{A}$ is the intersection of all closed sets containing A.*

5.2. PROPOSITION. *Let (X, d) be a metric space and $A \subset X$, then $\overline{A}$ is closed. Moreover, A is closed if and only if $A = \overline{A}$.*

It will be convenient to have another description of $\overline{A}$.

5.3. PROPOSITION. *Let (X, d) be a metric space and $A \subset X$, then if C is any closed set with $A \subseteq C$, then $\overline{A} \subseteq C$.*

Thus, $\overline{A}$ is the smallest closed set containing A.

5.4. EXAMPLE. *The closure of $(0,1)$ in $\mathbb{R}$, equipped with the usual metric, is $[0,1]$. Why? If E is closed and $(0,1) \subset E$, then $\{0,1\} \in E$, because $p_n = \frac{1}{1+n}$ defines a sequence $\{p_n\}$ with $p_n \in (0,1)$ for each n and $\lim_n p_n = 0$, and $q_n = 1 - p_n$ defines a sequence with limit $\lim_n q_n = 1$. So any closed set containing $(0,1)$ must also contain $[0,1]$, but $[0,1]$ is closed, and the smallest such set is $[0,1]$ itself.*

5.5. THEOREM. *Let $p \in X$, then $p \in \overline{A}$ if and only if for every $r > 0$, $B(p;r) \cap A$ is non-empty.*

5.6. EXAMPLE. *If $X = \{x \in \mathbb{R} : x > 0\}$ with the usual metric and $A = \{x : 0 < x < 1\}$ then $\overline{A} = (0, 1]$.*

By analogy, we define the interior of a set.

5.7. DEFINITION. *Let (X,d) be a metric space and $A \subseteq X$. The interior of A is defined by*

$$A^\circ = \cup\{E \subset A : E \text{ open}\}.$$

5.8. PROPOSITION. *Let (X, d) be a metric space and $A \subset X$, then A° is open. Moreover, A is open if and only if $A = A^\circ$.*

5.9. PROPOSITION. *Let (X, d) be a metric space and $A \subset X$, then*

$$A^\circ = \{x \in A : \text{ there is } \delta > 0 \text{ and } B(x, \delta) \subset A\}.$$

5.10. EXAMPLE. *Let $X = \mathbb{R}$ with the usual metric. Then $int([a, b]) = (a, b)$.*

5.11. EXAMPLE. *Let $X = \mathbb{R}^2$ with the Euclidean metric. Then $int(\{(x_1, x_2) : a \le x_1 \le b, c \le x_2 \le d\}) = \{(x_1, x_2) : a < x_1 < b, c < x_2 < d\}$ and $int(\{(x_1, 0) : a \le x_1 \le b\})$ is the empty set.*

5.12. EXAMPLE. *Let $\mathbb{R}$ be equipped with the usual metric and C be the Cantor set, then $C^\circ = \emptyset$.*

5.13. DEFINITION. *Let (X,d) be a metric space and let $A \subseteq X$. Then the* **boundary of** A, *denoted ∂A is the set $\partial A = \overline{A} \setminus A^\circ$.*

5.14. EXAMPLE. *Let $X = \mathbb{R}$ with the usual metric and let $A = (0, 1]$. Then $\partial A = \{0, 1\}$ and $\partial \mathbb{Q} = \mathbb{R}$.*

5.15. LEMMA. *In a metric space (X, d), for each $S \subset X$,*

$$\overline{S^c} = (S^\circ)^c.$$

5.16. PROPOSITION. *Given a metric space (X, d) and $S \subset X$, then $p \in \partial S$ if and only if for each $r > 0$, $B(p, r) \cap S \neq \emptyset$ and $B(p, r) \cap S^c \neq \emptyset$.*

5.17. COROLLARY. *Let $p \in X$. $p \in \partial S$ if and only if there is a sequence $\{p_n\} \subseteq S$ and a sequence $\{q_n\} \subseteq S^c$ with $\lim_n p_n = \lim_n q_n = p$.*

5.18. EXAMPLE. *Let $X = \mathbb{R}$, equipped with the usual metric, then $\overline{\mathbb{Q}} = \mathbb{R}$, $\mathbb{Q}^\circ = \emptyset$, so $\partial \mathbb{Q} = \mathbb{R}$.*

1.6. Completeness

One weakness of "convergence" is that when we want to prove that a sequence $\{p_n\}$ converges, then we need the point p that it converges to before we can prove that it converges. But often in math, one doesn't know yet that a problem has a "solution" and we can only produce a sequence $\{p_n\}$ that somehow is a better and better approximate solution and we want to claim that necessarily a point exists that *is* the limit of this sequence. It is for these reasons that mathematicians introduced the concepts of *Cauchy sequences* and *complete metric spaces.*

6.1. DEFINITION. *Let (X,d) be a metric space. A sequence* $\{p_n\} \subseteq X$ *is called* **Cauchy** *provided that for each* $\epsilon > 0$ *there exists* N *so that whenever* $m, n > N$, *then* $d(p_n, p_m) < \epsilon$.

We look at a few properties of Cauchy sequences.

6.2. PROPOSITION. *Let (X,d) be a metric space and let* $\{p_n\} \subseteq X$ *be a sequence. If* $\{p_n\}$ *is a convergent sequence, then* $\{p_n\}$ *is a Cauchy sequence.*

6.3. PROPOSITION. *Let (X,d) be a metric space,* $\{p_n\} \subseteq X$ *a sequence and* $\{p_{n_k}\}$ *a subsequence. If* $\{p_n\}$ *is Cauchy, then* $\{p_{n_k}\}$ *is Cauchy, i.e., every subsequence of a Cauchy sequence is also Cauchy.*

6.4. PROPOSITION. *Let (X,d) be a metric space and let* $\{p_n\} \subseteq X$ *be a Cauchy sequence. Then* $\{p_n\}$ *is bounded.*

6.5. DEFINITION. *Let (X,d) be a metric space. If for each Cauchy sequence in (X,d), there is a point in X that the sequence converges to, then (X,d) is called a* **complete metric space.**

6.6. EXAMPLE. *Let* $X = (0,1] \subseteq \mathbb{R}$ *be endowed with the usual metric* $d(x,y) = |x-y|$. *Then* X *is not complete, since* $\{\frac{1}{n}\} \subseteq X$ *is a Cauchy sequence with no point in* X *that it can converge to.*

6.7. EXAMPLE. *Let* $\mathbb{Q}$ *denote the rational numbers with metric* $d(x,y) = |x-y|$. *We can take a sequence of rational numbers converging to* $\sqrt{2}$, *which we know is irrational. Then that sequence will be Cauchy, but not have a limit in* $\mathbb{Q}$. *Thus,* $(\mathbb{Q}, d)$ *is not complete.*

In Math 3333, we proved that $\mathbb{R}$ with the usual metric has the property that every Cauchy sequence converges, that is, $(\mathbb{R}, d)$ is a complete metric space. This fact is so important that we repeat the proof here. First, we need to recall a few important facts and definitions.
Recall that a set $S \subseteq \mathbb{R}$ is called **bounded above** if there is a number $b \in \mathbb{R}$ such that $s \in S$ implies that $s \leq b$. Such a number b is called an **upper bound for S.** An upper bound for S that is smaller than every other upper bound of S is called a **least upper bound for S** and denoted **lub(S)** in Rosenlicht's book. In the book that we used for Math 3333, a least upper bound for S was called a **supremum for S** and denoted **sup(S).**
Similarly, a set $S \subseteq \mathbb{R}$ is called **bounded below** if there is a number $a \in \mathbb{R}$ such that $s \in S$ implies that $a \leq s$. Such a number a is called a **lower bound for S.** A lower bound for S that is larger than every other lower bound is called a **greatest lower bound for S,** and denoted **glb(S)** in Rosenlicht's book. In the book that we used for Math 3333, a greatest lower bound was called an **infimum for S** and denoted **inf(S).**

A key property of $\mathbb{R}$ is that every set that is bounded above has a least upper bound and that every set that is bounded below has a greatest lower bound.

6.8. PROPOSITION. *Let* $\{a_n\}$ *be a sequence of real numbers such that the set* $S = \{a_n : n \geq 1\}$ *is bounded above and such that* $a_n \leq a_{n+1}$ *for every* n. *Then* $\{a_n\}$ *converges and* $\lim_n a_n = lub(S)$. *If* $\{b_n\}$ *is a sequence of real numbers such that* $S = \{b_n : n \geq 1\}$ *is bounded below and* $b_n \geq b_{n+1}$, *then* $\{b_n\}$ *converges and* $\lim_n b_n = glb(S)$.

6.9. THEOREM. *Let* $(\mathbb{R}, d)$ *denote the real numbers with the usual metric. Then* $(\mathbb{R}, d)$ *is complete, i.e., every Cauchy sequence of real numbers converges.*

Another set of important examples of compelte metric spaces are the Euclidean spaces.

6.10. THEOREM. *Let* $(\mathbb{R}^k, d)$ *denote* k*-dimensional Euclidean space. Then* $(\mathbb{R}^k, d)$ *is complete.*

6.11. PROPOSITION. *Let* X *be a set and let* d *and* ρ *be two uniformly equivalent metrics on* X. *The metric space* (X, d) *is complete if and only if* (X, ρ) *is complete.*

By the above theorem and proposition, $(\mathbb{R}^k, d_1)$ and $(\mathbb{R}^k, d_\infty)$ are also complete metric spaces.

6.12. EXAMPLE. *Let* $X = (0, 1]$. *Recall that* X *is not complete in the usual metric* $d(x, y) = |x - y|$. *Given* $x, y \in X$ *we set* $\gamma(x, y) = |\frac{1}{x} - \frac{1}{y}|$. *It is easy to check that* γ *is a metric on* X. *We claim that* (X, γ) *is complete! Thus,* d *and* γ *are not uniformly equivalent.*

We sketch the proof. To prove this we must show that if $\{x_n\} \subseteq X$ is Cauchy in the γ metric, then it converges to a point in X. Given $\epsilon > 0$, suppose that for $n, m > N$, $\gamma(x_n, x_m) < \epsilon$. Since $\gamma(x_n, x_m) = |\frac{1}{x_n} - \frac{1}{x_m}| = |\frac{x_n - x_m}{x_n x_m}|$ we have that $|x_n - x_m| < \epsilon |x_n x_m| \leq \epsilon$. Thus, $\{x_n\}$ is Cauchy in the usual metric. Let $x = \lim_n x_n$. Since, $0 < x_n \leq 1$ we have that $0 \leq x \leq 1$. We claim that $x \neq 0$. Because if $x = 0$, then for any N, when $n, m > N$, if we fix m and we let $n \to +\infty$, $x_n \to 0$, then $\frac{1}{x_n} \to +\infty$. Thus, $\gamma(x_n, x_m) = |\frac{1}{x_n} - \frac{1}{x_m}| \to +\infty$. This prevents us making $\gamma(x_n, x_m) < \epsilon$ and so violates the Cauchy condition. Thus, $x \neq 0$. But also $x_n \neq 0$ for every n. By one of our basic results from 3333, when $x \neq 0, x_n \neq 0$ and $\lim_n x_n = x$, then $\lim_n \frac{1}{x_n} = \frac{1}{x}$. But this last limit being true means that for any $\epsilon > 0$, we can pick N so that when $n > N$, $|\frac{1}{x} - \frac{1}{x_n}| < \epsilon$. But this implies that for $n > N$, $\gamma(x, x_n) < \epsilon$ and so $\{x_n\}$ converges to x in the γ metric! We are done.
Now that we have a few examples of complete metric spaces, the following result gives us many more examples.

6.13. PROPOSITION. *Let (X,d) be a complete metric space. If* $Y \subseteq X$ *is a closed subset, then (Y,d) is a complete metric space.*

1.7. Compact Sets

7.1. DEFINITION. *Let (X,d) be a metric space, $S \subseteq X$. A collection $\{U_\alpha\}_{\alpha\in A}$ of subsets of X is called a* **cover of S** *provided that $S \subseteq \cup_{\alpha\in A}U_\alpha$ and an* **open cover of S** *provided that it is a cover of S and every set U_α is open. A subset $S \subseteq X$ is called* **compact** *provided that whenever $\{U_\alpha\}_{\alpha\in A}$ is an open cover of S, then there is a finite subset $F \subseteq A$ such that $S \subseteq \cup_{\alpha\in F}U_\alpha$. The collection $\{U_\alpha\}_{\alpha\in F}$ is called a* **finite subcover.**

7.2. EXAMPLE. *Let $\mathbb{R}$ have the usual metric. Let $U_n = B(0;n) = (-n,+n)$, $n \in \mathbb{N}$. Then these sets are open and $\mathbb{R} = \cup_{n\in\mathbb{N}}U_n$. Suppose that there was a finite subset $F = \{n_1,...,n_L\} \subseteq \mathbb{N}$ so that $\mathbb{R} \subseteq \cup_{n\in F}U_n = U_{n_1} \cup \cdots \cup U_{n_L}$. If we let $N = \max\{n_1,...,n_L\}$, then since $n < m$ implie that $U_n \subseteq U_m$, we would have that $\mathbb{R} \subseteq \cup_{n\in F}U_n = U_N$. But this implies that every real number is in $B(0;N)$, a contradiction. Hence, no finite subcover of $\{U_n\}_{n\in\mathbb{N}}$ covers $\mathbb{R}$ and so $\mathbb{R}$* **is not compact.**

7.3. PROPOSITION. *Let (X,d) be a discrete metric space and let $K \subseteq X$. Then K is compact if and only if K is a finite set.*

Before we can give many more examples of compact sets, we need some theorems.

7.4. PROPOSITION. *Let (X,d) be a metric space. If $K \subseteq X$ is compact, then K is closed.*

7.5. PROPOSITION. *Let (X,d) be a metric space, $K \subseteq X$ a compact set and let $S \subseteq K$ be a closed subset. Then S is compact.*

The next few results give a better understanding of the structure of compact sets.

7.6. PROPOSITION. *Let (X,d) be a metric space, $K \subseteq X$ a compact set and $S \subseteq K$ an infinite subset. Then S has a cluster point in K.*

Subsequences and compactness. We recall the definition of a subsequence.

7.7. DEFINITION. *Given a set X, a sequence $\{p_n\}$ in X and numbers, $1 \leq n_1 < n_2 < \cdots$, the new sequence that we get by setting $q_k = p_{n_k}$ is called a* **subsequence of** $\{p_n\}$.

For example, if $n_k = 2k$, $k = 1,2,...,$ then the subsequence that we get is just the even numbered terms of the old sequence. If $n_k = 2k-1$, $k = 1,2,...,$ then we get the subsequence of odd terms.
Often we simply denote the subsequence by $\{p_{n_k}\}$.

7.8. PROPOSITION. *Let (X,d) be a metric space, $\{p_n\}$ a sequence in X and $p \in X$. If the sequence $\{p_n\}$ converges to p, then every subsequence of $\{p_n\}$ also converges to p.*

7.9. DEFINITION. *Let (X,d) be a metric space, $K \subseteq X$. Then K is called* **sequentially compact** *if every sequence in K has a subsequence that converges to a point in K.*

In some texts a set is said to have the **Bolzano-Weierstrass property** if and only if it is sequentially compact.

7.10. PROPOSITION. *Let (X,d) be a metric space, $K \subseteq X$. If K is compact, then K is sequentially compact.*

7.11. DEFINITION. *Let (X,d) be a metric space, $K \subseteq X$ and let $\epsilon > 0$. A subset $E \subseteq K$ is called an* **ϵ-net for** *K provided that given any $p \in K$ there is $q \in E$, such that $d(p,q) < \epsilon$. The subset K is called* **totally bounded** *if for each $\epsilon > 0$ there is an ϵ-net for K consisting of finitely many points.*

7.12. EXAMPLE. *Let $K = [0,1]$ for each $\epsilon > 0$, let N be the largest integer so that $N\epsilon \leq 1$. Then $\{0, \epsilon, 2\epsilon, ..., N\epsilon\}$ is an ϵ-net for K.*

7.13. PROPOSITION. *Let (X,d) be a metric space and $K \subseteq X$. If K is sequentially compact, then K is totally bounded and complete.*

Now we come to the main theorem.

7.14. THEOREM. *Let (X,d) be a metric space and $K \subseteq X$. Then the following are equivalent:*

(1) *K is compact,*
(2) *K is sequentially compact,*
(3) *K is totally bounded and complete.*

7.15. THEOREM (Heine-Borel). *Let $(\mathbb{R}^k, d)$ denote k-dimensional Euclidean spaceand let $K \subseteq \mathbb{R}^k$. Then K is compact if and only if K is closed and bounded.*

7.16. THEOREM (Bolzano-Weierstrass). *Let $(\mathbb{R}^k, d)$ be k-dimensional space. If $\{p_n\} \subseteq \mathbb{R}^k$ is a bounded sequence, then it has a convergent subsequence.*

Historically, the Bolzano-Weierstrass theorem was proved before Heine-Borel. The original proof used a "divide and conquer" strategy.

7.17. DEFINITION. *Let (X,d) be a metric space and let $Y \subseteq X$. A subset $S \subseteq Y$ is called* **dense** *provided that $\overline{S} \cap Y = Y$. The set Y is called* **separable** *if there is a sequence $S = \{p_n\} \subseteq Y$ that is dense.*

7.18. PROPOSITION. *Let (X,d) be a metric space. If $K \subseteq X$ is compact, then K is separable.*