MATH 4331, PRACTICE FINAL EXAM, FALL 2013

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ABSTRACT. Write your name at the top! Put T beside each statement that is true, F beside each statement that is false. You will receive 10 points for each correct answer, no points for a wrong answer, 3 points for no answer.

TRUE-FALSE PROBLEMS

Throughout (X, d) and (Y, ρ) denote metric spaces and $\mathbb{R}$ is given the usual metric.

(1) If $f : X \to Y$ is continuous, and $U \subseteq Y$ is open, then $f^{-1}(U)$ is open.
(2) If $p \neq q$, then $d(p, q) \neq d(q, p)$.
(3) If U is an open set and C is a closed set, then $U \backslash C$ is open.
(4) If (X, d) is a metric space, $Y \subset X$ and (Y, d) is complete, then Y is closed.

For the remaining problems, all spaces are Euclidean spaces.

(1) If C is closed and U is open, then $U \backslash C$ is open.
(2) The Cantor set is compact.
(3) If $S = \{1 - \frac{1}{n} : n \in \mathbb{N}\}$, then $\sup S = 1$.
(4) If $S = [0, 1) \cup (1, 2] \subseteq \mathbb{R}$, then S is connected.

1. Problem(15 pts)

Let (X,d) be a metric space, let $Y \subseteq X$, and consider (Y, d) where d is restricted to pairs of points from Y. Give an example of sets $A \subseteq Y \subseteq X$ such that A is open in (Y,d) but A is not open in X. Prove that if Y is an open subset of X, then any set $A \subseteq Y$ that is open in (Y,d) is also open in (X,d).

2. Problem(20 pts)

Let (X, d) and (Y, ρ) be two metric spaces and $f : X \to Y$ and $g : Y \to \mathbb{R}$ be uniformly continuous. Show that $h : X \to \mathbb{R}$, $h(x) = g(f(x))$ is uniformly continuous.

3. Problem(20 pts)

Let X be a set and let d_1, d_2 be two metrics on X. Prove that the function $d(x, y) = d_1(x, y) + d_2(x, y)$ is a metric on X. Prove that if $C \subseteq X$ is a subset such that C is closed in either (X, d_1) or in (X, d_2) then C is closed in (X, d).

4. Problem(20 pts)

Let $\alpha_i : [a, b] \to \mathbb{R}, i = 1, 2$ be increasing, let $\alpha = \alpha_1 + \alpha_2$ and let $f : [a, b] \to \mathbb{R}$ be a bounded function that is Riemann-Stieltjes integrable on $[a, b]$ with respect to α_1 and α_2. Prove that f is Riemann-Stieltjes integrable on $[a, b]$ with respect to α.

5. Problem(20 pts)

Let $\mathbb{R}$ be endowed with the usual metric and let f be defined by

$$f(x) = \begin{cases} \frac{\sin(x)}{x} & x \neq 0 \\ 1 & x = 0 \end{cases}.$$

Prove that f is continuous at 0. You may use the inequality $\cos(x) \leq \sin(x)/x$ for $|x| \leq \pi$ and quote other results from class to support your argument.

6. Problem(20 pts)

Let (X, d) be a metric space and $E \subset X$. Show that $p \in E$ is an accumulation point of E if and only if $p \in \overline{E \setminus \{p\}}$.

7. Problem(15 pts)

State the Riemann integrability criterion. Show that if a bounded function f is Riemann integrable on $[0, 1]$, then $g(x) = |f(x)|$ is also Riemann integrable.

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