3.3. Application: Solution of ODE's

In this section we show how the contraction mapping principle can be used to deduce that solutions exist and are unique for some very complicated ordinary differential equations.
Starting with a continuous function $h(x, y)$ and an intial value y_0, we wish to solve

$$\frac{dy}{dx} = h(x, y), y(a) = y_0.$$

That is, we seek a function $f(x)$ so that $f'(x) = h(x, f(x))$ for $a < x < b$ and $f(a) = y_0$. Such a problem is often called an *initial value problem(IVP).*
For example, when $h(x, y) = x^2y^3$, then we are trying to solve, $\frac{dy}{dx} = x^2y^3$ which can be done by separation of variables. But our function could be $h(x, y) = sin(xy)$, in which case the differential equation becomes $\frac{dy}{dx} = sin(xy)$, which cannot be solved by elementary means.
For this application, we will use a number of things that we have not yet developed fully. But again, we stress that we are seeking motivation.
First note that by the fundamental theorem a continuous function $f : [a, b] \to \mathbb{R}$ satisfies the IVP if and only if for $a \le x \le b$,

$$f(x) = \int_a^x f'(t)dt + y_0 = \int_a^x h(t, f(t))dt + y_0.$$

This is often called the *integral form* of the IVP.
Now let $C([a, b])$ denote the set of all real-valued continuous functions on the interval $[a, b]$. Note that if we are given any $f \in C([a, b])$ and we set

$$g(x) = \int_a^x h(t, f(t))dt + y_0,$$

then g is also a continuous function on $[a, b]$.
Thus, we can define a map $\Phi : C([a, b]) \to C([a, b])$ by letting $\Phi(f) = g$, where f and g are as above. We see that solving our IVP is the same as finding a fixed point of the map Φ! Also, we are now in a situation, where by a "point" in our space $C([a, b])$, we mean a function.
This is starting to look like an application of the contraction mapping principle. For this we would first need a metric d on the set $C([a, b])$(again points in this metric space are functions!) so that $(C([a, b]), d)$ is a complete metric space and then we would need Φ to be a contraction mapping.
It turns out that there is such a metric on $C([a, b])$ and that for many functions h the corresponding map Φ is a contraction mapping.
First for the metric. Given any two functions $f, g \in C([a, b])$, we set

$$d(f, g) = \sup\{|f(x) - g(x)| : a \le x \le b\}.$$

This is the example that was introduced in the our first section on metric spaces. Note that since f, g are continuous functions on the compact metric space $[a, b]$, we have that the continuous function $f - g$ is bounded. Hence, the supremum is finite. Also, it is clear that $d(f, g) = 0$ if and only if

$f(x) = g(x)$ for all x, i.e., if and only if $f = g$. Note that $d(f, g) = d(g, f)$. Finally, if $f, g, h \in C([a, b])$, then

$$\begin{aligned} d(f,g) = \sup\{|f(x) - h(x) + h(x) - g(x)| : a \le x \le b\} \le \\ \sup\{|f(x) - h(x)| + |h(x) - g(x)| : a \le x \le b\} \le \\ \sup\{|f(x) - h(x)| : a \le x \le b\} + \sup\{|h(x) - g(x)| : a \le x \le b\} = \\ d(f,h) + d(h,g). \end{aligned}$$

Thus, we see that d is indeed a metric on $C([a, b])$. To apply the contraction mapping principle, we need this metric space to be complete. This fact is shown by the following theorem.

3.1. THEOREM. *The metric space* $(C([a, b]), d)$ *is complete.*

We now have all the tools at our disposal needed to solve some IVP's.

3.2. THEOREM. *Let* $h(x, y)$ *be a continuous function on* $[a, b] \times \mathbb{R}$ *and assume that* $|h(x, y_1) - h(x, y_2)| \leq K|y_1 - y_2|$ *with* $r = K(b - a) < 1$. *Then for any* y_0, *the initial value problem,* $f'(x) = h(x, f(x))$, $f(a) = y_0$ *has a unique solution on the interval* $[a, b]$.

The contraction mapping principle not only gives us the existence and uniqueness of solutions to the IVP, but it also gives us a method for approximating solutions and very nice bounds on the error of the approximation. The above theorem is far from the most useful, because the conditions on $h(x, y)$ are too restrictive for most applications. For example, $h(x, y) = x^3y^2$ doesn't satisfy our hypothesis. For this reason you will seldom see it in a textbook. But it is does have the advantage of being the simplest to prove and we believe that it illustrates the key guiding principles of the proofs of more complicated results.