CHAPTER 4

Riemann and Riemann-Stieltjes Integration

In this chapter we develop the theory of the Riemann integral, which is the type of integration used in your calculus courses and we also introduce Riemann-Stieltjes integration which is widely used in probability, statistics and financial mathematics.

4.1. The Riemann integral

Given a closed interval $[a, b]$ by a **partition** of $[a, b]$ we mean a set $\mathcal{P} = \{x_0.x_1, , ..., x_{n-1}, x_n\}$ with $a = x_0 < x_1 < ... < x_n = b$. The **norm** or **width** of the partition is

$$\|\mathcal{P}\| = \max\{x_i - x_{i-1} : 1 \leq i \leq n\}.$$

Given two partitions $\mathcal{P}_1$ and $\mathcal{P}_2$ we say that $\mathcal{P}_2$ is a **refinement of** $\mathcal{P}_1$ or $\mathcal{P}_2$ **refines** $\mathcal{P}_1$ provided that as sets $\mathcal{P}_1 \subseteq \mathcal{P}_2$. Note that if $\mathcal{P}_2$ refines $\mathcal{P}_1$, then $\|\mathcal{P}_2\| \leq \|\mathcal{P}_1\|$.
Given a bounded function $f : [a, b] \to \mathbb{R}$ and a partition $\mathcal{P} = \{x_0, ..., x_n\}$ of $[a, b]$ for $i = 1, ..., n$, we set

$$M_i = \sup\{f(x) : x_{i-1} \leq x \leq x_i\}$$

and

$$m_i = \inf\{f(x) : x_{i-1} \leq x \leq x_i\}.$$

The **upper Riemann sum of f** given the partition $\mathcal{P}$ is the real number,

$$U(f, \mathcal{P}) = \sum_{i=1}^{n} M_i(x_i - x_{i-1}),$$

and the **lower Riemann sum of f** is

$$L(f, \mathcal{P}) = \sum_{i=1}^{n} m_i(x_i - x_{i-1}).$$

Note that if we hadn't assumed that f is a bounded function then some of the numbers M_i or m_i would have been infinite. This is the one reason that we can only define Riemann integrals for bounded functions.
By a **general Riemann sum of f given** $\mathcal{P}$, we mean a sum of the form

$$\sum_{i=1}^{n} f(x_i')(x_i - x_{i-1}),$$

where x_i' is any choice of points satisfying, $x_{i-1} \leq x_i' \leq x_i$, for $i = 1, ..., n$. Since $m_i \leq f(x_i') \leq M_i$, the upper and lower Riemann give an upper and lower bound for general Riemann sums, i.e.,

$$L(f, \mathcal{P}) \leq \sum_{i=1}^{n} f(x_i')(x_i - x_{i-1}) \leq U(f, \mathcal{P}).$$

In fact, since we can choose the points x_i' so that $f(x_i')$ is arbitrarily close to M_i, we see that $U(f, \mathcal{P})$ is actually the supremum of all general Riemann sums of f given $\mathcal{P}$. Similarly, by choosing the points x_i' so that $f(x_i')$ is arbitrarily close to m_i, we see that $L(f, \mathcal{P})$ is the infimum of all general Riemann sums.

Thus, if we want all general Riemann sums of a function to be "close" to a value that we wish to think of as the "integral of f", then it will be enough to study the "extreme" cases of the upper and lower sums.

1.1. PROPOSITION. *Let $f : [a, b] \to \mathbb{R}$ be a bounded function and let $\mathcal{P}_1$ and $\mathcal{P}_2$ be partitions of $[a, b]$ with $\mathcal{P}_2$ a refinement of $\mathcal{P}_1$. Then*

$$L(f, \mathcal{P}_1) \leq L(f, \mathcal{P}_2) \leq U(f, \mathcal{P}_2) \leq U(f, \mathcal{P}_1).$$

1.2. DEFINITION. *Let $f : [a,b] \to \mathbb{R}$ be a bounded function. Then the* **upper Riemann integral of f** *is the number*

$$\overline{\int_a^b} f(x)dx = \inf\{U(f,\mathcal{P}) : \mathcal{P} \text{ a partition of } [a,b]\}.$$

The **lower Riemann integral of f** *is the number*

$$\underline{\int_a^b} f(x)dx = \sup\{L(f,\mathcal{P}) : \mathcal{P} \text{ a partition of } [a,b]\}.$$

We say that f is **Riemann integrable** *when these two numbers are equal and in this case we define the* **Riemann integral of f** *to be*

$$\int_a^b f(x)dx = \overline{\int_a^b} f(x)dx = \underline{\int_a^b} f(x)dx.$$

To help cement these definitions, let us consider the function, $f : [a,b] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 1 & x \text{ rational} \\ 0 & x \text{ irrational}, \end{cases}$$

then for any partition $\mathcal{P}$ we will have that $M_i = 1$ and $m_i = 0$ for every i. Hence, $U(f,\mathcal{P}) = (b-a)$ and $L(f,\mathcal{P}) = 0$. Thus,

$$\overline{\int_a^b} f(x)dx = b - a \text{ and } \underline{\int_a^b} f(x)dx = 0.$$

In particular, f is not Riemann integrable.
The following helps to explain the terms "upper" and "lower".

1.3. PROPOSITION. *Let $f : [a,b] \to \mathbb{R}$ be a bounded function and let $\mathcal{P}_1$ and $\mathcal{P}_2$ be any two partitions of $[a,b]$. Then $L(f,\mathcal{P}_1) \le U(f,\mathcal{P}_2)$ and $\underline{\int_a^b} f(x)dx \le \overline{\int_a^b} f(x)dx$.*

The following gives an important means of determining if a function is Riemann integrable.

1.4. THEOREM. *Let $f : [a, b] \to \mathbb{R}$ be a bounded function. Then f is Riemann integrable if and only if for every $\epsilon > 0$ there exists a partition $\mathcal{P}$ such that $U(f, \mathcal{P}) - L(f, \mathcal{P}) < \epsilon$.*

1.5. DEFINITION. *We say that a bounded function* $f : [a, b] \to \mathbb{R}$ *satisfies the* **Riemann integrability criterion** *provided that for every* $\epsilon > 0$, *there exists a partition* $\mathcal{P}$, *such that* $U(f, \mathcal{P}) - L(f, \mathcal{P}) < \epsilon$.

Thus, the above theorem says that a bounded function is Riemann integrable if and only if it satisfies the Riemann integrability criterion.

1.6. THEOREM. *Let* $f : [a, b] \to \mathbb{R}$ *be a continuous function. Then* f *is Riemann integrable.*

1.7. PROBLEM. *Let $f(x) = x$. Compute $U(f, \mathcal{P}_n)$ and $L(f, \mathcal{P}_n)$. Use these formulas and the Riemann integrability criterion to prove that f is Riemann integrable on $[0,1]$ and to prove that $\int_0^1 x dx = 1/2$.*

1.8. PROBLEM. *Let $a \leq c < d \leq b$ and let $f : [a,b] \to \mathbb{R}$ be the function defined by*

$$f(x) = \begin{cases} 0 & a \leq x \leq c \\ 1 & c < x < d \\ 0 & d \leq x \leq b \end{cases}.$$

Prove that f is Riemann integrable on $[a,b]$ and that $\int_a^b f(x)dx = (d-c)$.

4.2. The Riemann-Stieltjes Integral

The Riemann-Stieltjes integral is a slight generalization of the Riemann integral. The new ingredient in Riemann-Stieltjes integration is a function,

$$\alpha : [a,b] \to \mathbb{R}$$

that is increasing, i.e., $x \leq y$ implies that $\alpha(x) \leq \alpha(y)$. It is best to think of α as a function that measures a new "length" of subintervals by setting the length of a subinterval $[x_{i-1}, x_i]$ equal to $\alpha(x_i) - \alpha(x_{i-1})$. One case where this concept arises is if we imagine that we have a piece of wire of varying density stretched from a to b and $\alpha(x_i) - \alpha(x_{i-1})$ represents the weight of the section of wire from x_{i-1} to x_i.
Given a bounded function $f : [a,b] \to \mathbb{R}$ the Riemann-Stieltjes integral is denoted

$$\int_a^b f d\alpha,$$

and it is designed to also define a " signed area" under the graph of f but now if we want the area of a rectangle to be the length of the base times the height, then a rectangle from x_{i-1} to x_i of height h should have area

$$h(\alpha(x_i) - \alpha(x_{i-1})).$$

Thus, given a bounded function f an increasing function α, a partition $\mathcal{P} = \{a = x_0, ..., x_n = b\}$, the numbers $M_i = \sup\{f(x) : x_{i-1} \leq x \leq x_i\}$ and $m_i = \inf\{f(x) : x_{i-1} \leq x \leq x_i\}$, we are led to define the **upper Riemann-Stieltjes sum** as

$$U(f, \mathcal{P}, \alpha) = \sum_{i=1}^{n} M_i(\alpha(x_i) - \alpha(x_{i-1}))$$

and the **lower Riemann-Stieltjes sum** as

$$L(f, \mathcal{P}, \alpha) = \sum_{i=1}^{n} m_i(\alpha(x_i) - \alpha(x_{i-1})).$$

The **upper Riemann-Stieltjes integral of f with respect to α** is then defined to be

$$\overline{\int_a^b} f d\alpha = \inf\{U(f, \mathcal{P}, \alpha) : \mathcal{P} \text{ is a partition of } [a, b]\}.$$

Similarly, the **lower Riemann-Stieltjes integral of f with respect to α** is defined to be

$$\underline{\int_a^b} f d\alpha = \sup\{L(f, \mathcal{P}, \alpha) : \mathcal{P} \text{ is a partition of } [a, b]\}.$$

When

$$\overline{\int_a^b} f d\alpha = \underline{\int_a^b} f d\alpha,$$

then we say that f **is Riemann-Stieltjes integrable with respect to α** and we let

$$\int_a^b f d\alpha$$

denote this common value.

We repeat the key facts about Riemann-Stieltjes integration below. Since the proofs are almost identical to the corresponding proofs in the case of Riemann integration, we omit the details.

2.1. PROPOSITION. *Let $f : [a, b] \to \mathbb{R}$ be a bounded function, let $\alpha : [a, b] \to \mathbb{R}$ be an increasing function and let $\mathcal{P}_1$ and $\mathcal{P}_2$ be partitions of $[a, b]$ with $\mathcal{P}_2$ a refinement of $\mathcal{P}_1$. Then*

$$L(f, \mathcal{P}_1, \alpha) \leq L(f, \mathcal{P}_2, \alpha) \leq U(f, \mathcal{P}_2, \alpha) \leq U(f, \mathcal{P}_1, \alpha).$$

2.2. PROPOSITION. *Let* $f : [a,b] \to \mathbb{R}$ *be a bounded function, let* $\alpha : [a,b] \to \mathbb{R}$ *be increasing and let* $\mathcal{P}_1$ *and* $\mathcal{P}_2$ *be any two partitions of* $[a,b]$. *Then* $L(f,\mathcal{P}_1,\alpha) \leq U(f,\mathcal{P}_2,\alpha)$ *and* $\underline{\int_a^b} f(x)d\alpha \leq \overline{\int_a^b} f(x)d\alpha$.

2.3. THEOREM. *Let* $f : [a, b] \to \mathbb{R}$ *be a bounded function and let* $\alpha : [a, b] \to \mathbb{R}$ *be increasing. Then* f *is Riemann-Stieltjes integrable with respect to* α *if and only if for every* $\epsilon > 0$ *there exists a partition* $\mathcal{P}$ *such that* $U(f, \mathcal{P}, \alpha) - L(f, \mathcal{P}, \alpha) < \epsilon$.

2.4. DEFINITION. *Given an increasing function* $\alpha : [a, b] \to \mathbb{R}$, *we say that a bounded function* $f : [a, b] \to \mathbb{R}$ *satisfies the* **Riemann-Stieltjes integrability criterion with respect to** α *provided that for every* $\epsilon > 0$, *there exists a partition* $\mathcal{P}$, *such that* $U(f, \mathcal{P}, \alpha) - L(f, \mathcal{P}, \alpha) < \epsilon$.

Thus, the above theorem says that a bounded function is Riemann-Stieltjes integrable with respect to α if and only if it satisfies the Riemann-Stieltjes integrability criterion with respect to α.

2.5. THEOREM. *Let* $f : [a, b] \to \mathbb{R}$ *be a continuous function and let* $\alpha : [a, b] \to \mathbb{R}$ *be an increasing function. Then* f *is Riemann-Stieltjes integrable with respect to* α.