One of the main motivations for Riemann-Stieltjes integration comes from the concept of a *cumulative distribution function of a random variable.* To understand the Riemann-Stieltjes integral, one need not understand any probability theory, but we introduce these ideas from probability here, via an example, in order to motivate the desire for the Riemann-Stieltjes integral. For a *probability space,* suppose that we consider the flip of a "biased" coin, so tht the probability of heads(H) is p and the probability of tails(T) is $1-p$ with $0 < p < 1$. When $p = 1/2$, we think of the coin as a "fair" coin. A *random variable* would then be a function that assigned a real number to each of these possible outcomes. For example, we could define a random variable X by setting $X(H) = 1$, $X(T) = 3$. If we let $Prob(X = x)$ denote the probability that X is equal to the real number x, then we would have $Prob(X = 1) = p$ and $Prob(X = 3) = 1 - p$. The *cumulative distribution function of* X, is the function $Prob(X \leq x)$. So in our case,

$$Prob(X \leq x) = \begin{cases} 0 & x < 1 \\ p & 1 \leq x < 3 \\ 1 & 3 \leq x \end{cases}.$$

So our cumulative distribution function has two "jump" discontinuities, the first of size p ocurring at 1 and the second of size $1 - p$ occurring at 3.
More generally, if one has a probability space, a random variable X, and we let $\alpha(x) = Prob(X \leq x)$, then α will be an increasing function, i.e., one for which we can consider Riemann-Stieltjes integrals. Much work in probability and its applications, e.g., financial math, is concerned with computing these Riemann-Stieltjes integrals for various functions f in the case that α is the cumulative distribution function of a random variable.
Returning to our example, if for any c we let

$$J_c(x) = \begin{cases} 0 & x < c \\ 1 & c \leq x \end{cases},$$

then we can also write $Prob(X \leq x) = pJ_1(x) + (1-p)J_3(x)$.
These simple "jump" functions are useful for expressing the cumulative distributions of many random variables. For example, if we consider one roll of a "fair" die, so that each side has probability $1/6$ of facing up and we let X be the random variable that simply gives the number that is facing up, then

$$Prob(X \leq x) = 1/6[J_1(x) + J_2(x) + J_3(x) + J_4(x) + J_5(x) + J_6(x)].$$

For this reason we want to take a careful look at what Riemann-Stieltjes integration is when $\alpha(x) = hJ_c(x)$, for some real number c and $h > 0$.

2.6. DEFINITION. *Let $f : [a, b] \to \mathbb{R}$ and let $a < c \leq b$. If there is a number L so that for every $\epsilon > 0$, there exists $\delta > 0$, so that when $c - \delta < x < c$, we have $|f(x) - L| < \epsilon$, then we say that* **L is the limit from the left of f at c.** *We write $\lim_{x \to c^-} f(x) = L$ and say that f* **is continuous from**

the left at c *provided that* $\lim_{x\to c^-} f(x) = f(c)$. *When* $a \le c < b$, *then the* **limit from the right of** f **at** c *is defined similarly and is denoted* $\lim_{x\to c^+} f(x)$. *We say that* f **is continuous from the right at** c *provided that* $\lim_{x\to c^+} f(x) = f(c)$.

2.7. THEOREM. *Let* $a < c \le b$, *let* $h > 0$, *let* $\alpha(x) = hJ_c(x)$, *and let* $f : [a,b] \to \mathbb{R}$ *be a bounded function. Then* f *is Riemann-Stieltjes integrable with respect to* α *if and only if* f *is continuous from the left at* c. *In this case,*

$$\int_a^b f d\alpha = hf(c).$$

In probability, two events are called independent if the probability of both occurring is the product of the probabilities of each occurring. For example, suppose that we flip a biased coin with $Prob(H) = p, Prob(T) = 1-p$ twice, so that the possible outcomes are $\{HH, HT, TH, TT\}$. If we assume that the flips are independent then the outcomes would have probabilities,

$$Prob(HH) = Prob(H)Prob(H) = p^2,$$
$$Prob(HT) = Prob(TH) = Prob(H)Prob(T) = p(1-p),$$
$$Prob(TT) = Prob(T)Prob(T) = (1-p)^2.$$

4.3. Properties of the Riemann-Stieltjes Integral

Before trying to compute many integrals, it will be helpful to have proven some basic properties of these integrals.

3.1. THEOREM. *Let* $\alpha : [a,b] \to \mathbb{R}$ *be an increasing function, let* $f, g : [a,b] \to \mathbb{R}$ *be bounded functions that are Riemann-Stieltjes integrable with respect to* α *and let* $c \in \mathbb{R}$ *be a constant. Then:*

(1) cf *is Riemann-Stieltjes integrable with respect to* α *and*

$$\int_a^b cf d\alpha = c \int_a^b f d\alpha,$$

(2) $f + g$ *is Riemann-Stieltjes integrable with respect to* α *and*

$$\int_a^b (f+g) d\alpha = \int_a^b f d\alpha + \int_a^b g d\alpha,$$

(3) fg *is Riemann-Stieltjes integrable with respect to* α.

3.2. PROPOSITION. *Let* $\alpha_1, \alpha_2 : [a,b] \to \mathbb{R}$ *be increasing functions, let* $c > 0$ *be a constant and let* $f : [a,b] \to \mathbb{R}$ *be a bounded function. Then*

$$\underline{\int_a^b} f d(c\alpha_1 + \alpha_2) = c \underline{\int_a^b} f d\alpha_1 + \underline{\int_a^b} f d\alpha_2$$

and

$$\overline{\int_a^b} f d(c\alpha_1 + \alpha_2) = c \overline{\int_a^b} f d\alpha_1 + \overline{\int_a^b} f d\alpha_2.$$

3.3. PROPOSITION. *Let* $\alpha_1, \alpha_2 : [a,b] \to \mathbb{R}$ *be increasing, let* $c > 0$ *and let* $f : [a,b] \to \mathbb{R}$ *be a bounded function. If* f *is Riemann-Stieltjes integrable with respect to* α_1 *and* α_2*, then* f *is Riemann-Stieltjes integrable with respect to* $c\alpha_1 + \alpha_2$ *and*

$$\int_a^b f d(c\alpha_1 + \alpha_2) = c \int_a^b f d\alpha_1 + \int_a^b f d\alpha_2.$$

3.4. THEOREM. *Let* $a < c_1 < .... < c_n \leq b$, *let* $h_i > 0, i = 1, ..., n$ *and let* $\alpha = h_1 J_{c_1} + \cdots h_n J_{c_n}$. *If* $f : [a, b] \to \mathbb{R}$ *is a bounded function that is continuous from the left at each* $c_i, i = 1, ..., n$, *then* f *is Riemann-Stieltjes integrable with respect to* α *and*

$$\int_a^b f d\alpha = h_1 f(c_1) + \cdots h_n f(c_n).$$

4.4. The Fundamental Theorem of Calculus

Before proceeding it will be useful to have a few more facts about the Riemann-Stieltjes integral.

4.1. THEOREM (Mean Value theorem for Integrals). *Let* $\alpha : [a, b] \to \mathbb{R}$ *be an increasing function and let* $f : [a, b] \to \mathbb{R}$ *be a continuous function. Then there exists* $c, a \leq c \leq b$ *such that*

$$\int_a^b f d\alpha = f(c)(\alpha(b) - \alpha(c)).$$

4.2. PROPOSITION. *Let* $a \leq c < d \leq b$, *let* $\alpha : [a, b] \to \mathbb{R}$ *be an increasing function and let* $f : [a, b] \to \mathbb{R}$ *be a bounded function that is Riemann-Stieltjes integrable with respect to* α *on* $[a, b]$, *then* f *is Riemann-Stieltjes integrable with respect to* α *on* $[c, d]$.

In the case that $\alpha(x) = x$ the following theorem reduces to the classic fundamental theorem of calculus.

4.3. THEOREM (Fundamental Theorem of Calculus). *Let* $\alpha : [a, b] \to \mathbb{R}$ *be an increasing function that is differentiable on* (a, b) *and let* $f : [a, b] \to \mathbb{R}$ *be continuous. If* $F(x) = \int_a^x f(t)d\alpha(t)$, *then* F *is differentiable on* (a, b) *and* $F'(x) = f(x)\alpha'(x)$.

4.4. COROLLARY. *Let* $\alpha : [a, b] \to \mathbb{R}$ *be an increasing, continuous function that is differentiable on* (a, b) *and let* $f : [a, b] \to \mathbb{R}$ *be continuous. If* $F : [a, b] \to \mathbb{R}$ *is a continuous function that is differentiable on* (a, b) *and* $F'(x) = f(x)\alpha'(x)$, *then* $\int_a^b f d\alpha = F(b) - F(a)$.